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Collinear and Regge behavior of $2 \to 4$ MHV amplitude in $\mathcal{N} = 4$ super Yang-Mills theory

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Abstract

We investigate the collinear and Regge behavior of the $2 \to 4$ MHV amplitude in $\mathcal{N} = 4$ super Yang-Mills theory in the BFKL approach. The expression for the remainder function in the collinear kinematics proposed by Alday, Gaiotto, Maldacena, Sever and Vieira is analytically continued to the Mandelstam region. The result of the continuation in the Regge kinematics shows an agreement with the BFKL approach up to to five-loop level. We present the Regge theory interpretation of the obtained results and discuss some issues related to a possible non-multiplicative renormalization of the remainder function in the collinear limit.

1 Introduction

The recent developments in the study of the Maximally Helicity Violating (MHV) amplitudes in $\mathcal{N}=4$ super Yang-Mills theory encourage us to apply a well studied Balitsky-Fadin-Kuraev-Lipatov (BFKL) approach to test some analytic results available on the market. The study of the MHV amplitudes is traced back to the paper of Parke and Taylor [1], who showed that a tree-level gluon scattering amplitude significantly simplifies for a definite helicity configuration of the external particles. The simplicity of the tree MHV amplitudes suggested that they could have some nice properties also at the quantum level. This idea led to a formulation of the Anastasiou-Bern-Dixon-Kosower (ABDK) [2] and later to the Bern-Dixon-Smirnov (BDS) [3] all-loop formula for multi-leg MHV amplitudes in $\mathcal{N}=4$ SYM. The BDS ansatz was tested in the regimes of the strong coupling by Alday and Maldacena [4] and the weak coupling by two of the authors in collaboration with Sabio Vera [5]. Both of the studies showed some inconsistency of the BDS formula for a number of the external gluons being larger than five. At strong coupling the multi-leg MHV amplitude was considered in the limit of the very large number of external legs using the minimal surface approach [6]. At weak coupling, the analytic structure of the BDS amplitude was studied at two loops for four, five and six external gluons in the multi-Regge kinematics [5]. The BDS amplitude with four and five external gluons were shown to be compatible with the dispersive representation in the Regge kinematics, while the six gluon BDS amplitude at two loops could not match a form expected from the Regge theory. This deficiency becomes especially clear if we consider a physical kinematic region, where some of the energies are negative (this region has been named Mandestam region). It was argued [5, 7] that the BDS amplitude should be corrected starting at two loops and six external gluons due to the fact that it does not account properly for the so-called Regge or Mandelstam cuts in the complex angular momenta plane. The two loop correction to the six gluon BDS amplitude was calculated in the multi-Regge kinematics by two of the authors in collaboration with Sabio Vera [7] using the Balitsky-Fadin-Kuraev-Lipatov (BFKL) approach [8].

On the other hand recent studies showed an intimate relation between expectation value of polygon Wilson loops and scattering amplitudes in $\mathcal{N}=4$ SYM. It was assumed [6] that the BDS formula can be corrected by a multiplicative function named *the remainder function*, which depends only on conformal invariants (anharmonic ratios) in the dual momenta space [9, 10]. The remainder function for the six-gluon MHV amplitude was calculated by Drummond, Henn, Korchemsky and Sokatchev [11] and presented in terms of rather complicated four-fold integrals, which were simplified in the quasi-multi-Regge kinematics by Del Duca, Duhr and Smirnov [12, 13] and expressed in terms of generalized Goncharov polylogarithmic functions of three dual conformal cross ratios. Their result was greatly simplified by Goncharov, Spradlin, Vergu and Volovich (GSVV) [14] using the theory of motives and was compactly written in terms of only classical polylogarithms. The analytic continuation of the GSVV remainder function to the Mandelstam region in the multi-Regge kinematics was performed by two of the authors [15, 16] reproducing the leading logarithmic prediction of ref. [7]. It also confirmed [5, 16] the validity of the dispersion-like relations for the remainder function in the multi-Regge kinematics found by one of the authors [17] for the $2 \to 4$ and $3 \to 3$ scattering amplitudes. The six-particle MHV amplitude at the strong coupling was also investigated by one of the authors in collaboration with Kotanski and Schomerus [18] in the Mandelstam region in the multi-Regge kinematics. The analysis of the analytic properties of the system of Y-equations allowed to extract the leading asymptotics, which is related to the Pomeron intercept at strong coupling.

Besides the multi-Regge regime, the remainder function was also considered in the so-called *collinear kinematics*, where two or more external gluons become collinear. In this kinematics the remainder function vanishes, but subleading corrections can provide some information on anomalous dimensions of composite operators in the Operator Product Expansion (OPE) of the polygonal Wilson loops. The OPE analysis suggested by Alday, Gaiotto, Maldacena, Sever and

Vieira (AGMSV) [19] allowed to make prediction for the collinear behavior of the remainder function at strong and weak coupling in the Euclidean kinematics. This analysis was extended by Gaiotto, Maldacena, Sever and Vieira [20] to reproduce the full two remainder function of the six-particle MHV amplitude. It should, however, be kept in mind that this OPE expansion might be quite different from the usual short distance of light cone expansions of color singlet operators. Strictly speaking, in the present case we are dealing with planar amplitudes and all exchange channels are in adjoint color states; furthermore, there could be a non-multiplicative renormalization, i.e. one can have several operators with different anomalous dimensions.

In the present study we investigate the AGMSV expression for the remainder function for the six-gluon MHV amplitude at weak coupling and compare it with the BFKL predictions in the double-logarithmic approximation. We perform analytic continuation of the $2 \to 4$ amplitude to the Mandelstam region and extract the leading logarithmic terms in the multi-Regge kinematics reproducing the BFKL result up to five loops. In order to find this agreement we split the anomalous dimension given in the AGMSV formula into two pieces, each of them having poles only in one semiplane. We find that all the known BFKL contributions come only from one of these two contributions. This agrees with the Regge theory expectation to have a clear separation between the negative and positive poles for the s-channel discontinuities of the remainder function, suggesting a sum of two exponentiations of the anomalous dimensions, which becomes important already at three loops. The proposed alternative exponentiation agrees with the Regge theory analysis and coincides with the AGMSV expression at two loops. The ambiguity between the two exponentiations can be resolved by taking into account next-to-leading corrections to the eigenvalue of the BFKL Kernel in the adjoint representation, which are currently not available and will be calculated in the near future.

The content of the paper is presented as follows. In the first section we overview the BFKL analysis applied to the $2 \to 4$ scattering MHV amplitude in the multi-Regge kinematics. The section 3 is devoted to the collinear behavior of the remainder function in the Mandelstam region, where we calculate the all-loop expression in the collinear and multi-Regge kinematics with double logarithmic accuracy. Then we present details of the analytic continuation of the AGMSV remainder function to the Mandelstam channel and comparison with the BFKL approach up to five loops. In the section 4 we consider the interpretation of the obtained result from the point of view of the Regge theory and propose an alternative exponentiation for the anomalous dimension. The main results are discussed in the last section. Some detailed calculations are presented in the appendices.

2 Regge limit

In this section we discuss the (multi-) Regge kinematics of the six-gluon scattering MHV amplitude, considered in our previous studies in the regime of the weak [5, 7, 15, 16, 21] and strong coupling [18]. The six-gluon amplitude describes to two physical scattering processes, namely to $2 \to 4$ and $3 \to 3$ scattering. In the present study we are mainly interested in the $2 \to 4$ MHV amplitude at weak coupling in the physical channel, where the Mandelstam cuts give a non-vanishing contribution. We call the corresponding channels - the Mandelstam channels. For the purpose of the present discussion it is convenient to introduce the kinematic invariants shown in Fig. 1.

The invariants are defined as $s = (p_A + p_B)^2$, $s_1 = (p_{A'} + k_1)^2$, $s_2 = (k_1 + k_2)^2$, $s_3 = (p_{B'} + k_2)^2$, $s_{012} = (p_{A'} + k_1 + k_2)^2$, $s_{123} = (p_{B'} + k_1 + k_2)^2$, $t_1 = (p_A - p_{A'})^2$, $t_2 = (p_A - p_{A'} - k_1)^2$ and $t_3 = (p_B - p_{B'})^2$. The dual conformal cross ratios are given by

$$u_1 = \frac{s s_2}{s_{012}\, s_{123}},\ u_2 = \frac{s_1 t_3}{s_{012}\, t_2},\ u_3 = \frac{s_3 t_1}{s_{123} t_2}. \tag{1}$$

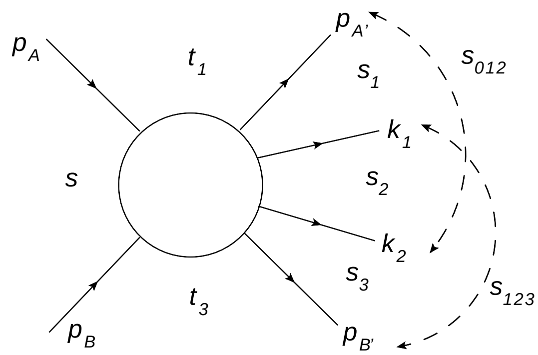

Figure 1: The $2 \to 4$ gluon scattering amplitude.

The multi-Regge kinematics, where $s \gg s_{012}\,, s_{123} \gg s_1,\; s_2,\; s_3 \gg |t_1|,\; |t_2|,\; |t_3|$ implies

$$1-u_1 \to +0, \;\; u_2 \to +0, \;\; u_3 \to +0, \;\; \frac{u_2}{1-u_1} \simeq \mathcal{O}(1), \;\; \frac{u_3}{1-u_1} \simeq \mathcal{O}(1), \tag{2}$$

which suggests that in this kinematics the convenient variables for the remainder function are $1-u_1$ and the reduced cross ratios defined by

$$\tilde{u}_2 = \frac{u_2}{1-u_1}, \;\; \tilde{u}_3 = \frac{u_3}{1-u_1}. \tag{3}$$

In the Regge limit they can be expressed through s_2 and the transverse momenta

$$1-u_1 \simeq \frac{(\mathbf{k}_1+\mathbf{k}_2)^2}{s_2}, \;\; \tilde{u}_2 \simeq \frac{\mathbf{k}_1^2\,\mathbf{q}_3^2}{(\mathbf{k}_1+\mathbf{k}_2)^2\,\mathbf{q}_2^2}, \;\; \tilde{u}_3 \simeq \frac{\mathbf{k}_2^2\,\mathbf{q}_1^2}{(\mathbf{k}_1+\mathbf{k}_2)^2\,\mathbf{q}_2^2}, \tag{4}$$

so that the energy s_2 dependence of the remainder function is related only to a dependence on u_1 and not on $\tilde{u}_2$ and $\tilde{u}_3$. This is not the only choice for expressing the energy dependence in terms of the dual cross ratios, but we do not consider other choices for the sake of clarity of the presentation.

In the "Euclidean" kinematics ($s,\; s_2 < 0$) the remainder function vanishes as it follows from the analysis presented in refs. [5, 7]. However, these studies also show that this is not the case in a slightly different physical region, where one or more dual conformal cross ratios possess a phase. This happens when some energy invariants change the sign. In the present paper we consider one of such regions of the $2 \to 4$ scattering amplitude having

$$u_1 = |u_1| e^{-i2\pi}, \tag{5}$$

together with u_2 and u_3 held fixed and positive. This corresponds to a physical region (the Mandelstam channel), where

$$s\,, s_2 > 0; \quad s_1,\; s_3,\; s_{012},\; s_{123} < 0 \tag{6}$$

as illustrated in Fig. 2. It is worth emphasizing that the scattering amplitude in Fig. 2 is still planar, but the produced particles have reversed momenta k_1 and k_2 with negative energy components.

In the Mandelstam channel the remainder function grows with energy and was first calculated using the BFKL approach by two of the authors in collaboration with A. Sabio Vera in ref. [7]. The BFKL approach, based on the analyticity and unitarity was developed more than thirty years ago [8]. In this approach one sums the contributions from the Feynman diagrams, which are enhanced by the logarithms of the energy ($1-u_1 \simeq (\mathbf{k}_1+\mathbf{k}_2)^2/s_2$ in our case). The Leading

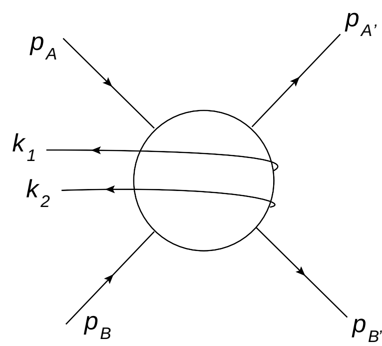

Figure 2: The Mandelstam channel of the $2 \to 4$ gluon planar scattering amplitude.

Logarithmic Approximation (LLA) allows to write an integral representation of the remainder function R^{LLA}_{BFKL} to any order of the parameter $g^2 \ln s_2$. The amplitude in this Mandelstam channel is given by [7]

$$M_{2\to 4} = M^{BDS}_{2\to 4}\, R_{BFKL} = M^{BDS}_{2\to 4}\,(1 + i\Delta_{2\to 4}), \tag{7}$$

where $M^{BDS}_{2\to 4}$ is the BDS expression [3] and the correction $\Delta_{2\to 4}$ was calculated in all orders with a leading logarithmic accuracy using the solution to the BFKL eigenvalue in the adjoint representation. The all-order LLA expression for $\Delta_{2\to 4}$ reads

$$\Delta^{LLA}_{2\to 4} = \frac{a}{2} \sum_{n=-\infty}^{\infty} (-1)^n \int_{-\infty}^{\infty} \frac{d\nu}{\nu^2 + \frac{n^2}{4}} \left(\frac{q_3^* k_1^*}{k_2^* q_1^*}\right)^{i\nu - \frac{n}{2}} \left(\frac{q_3 k_1}{k_2 q_1}\right)^{i\nu + \frac{n}{2}} \left(s_2^{\omega(\nu,n)} - 1\right) \tag{8}$$

$$\simeq \frac{a}{2} \sum_{n=-\infty}^{\infty} (-1)^n \int_{-\infty}^{\infty} \frac{d\nu}{\nu^2 + \frac{n^2}{4}} \,(w^*)^{i\nu - \frac{n}{2}}\, (w)^{i\nu + \frac{n}{2}} \left((1-u_1)^{-\omega(\nu,n)} - 1\right)$$

Here k_1, k_2 are complex transverse components of the gluon momenta, q_1, q_2, q_3 are the corresponding momenta of reggeons in the crossing channels. It is convenient to define holomorphic and antiholomorphic variables in the transverse space as

$$w = \frac{q_3 k_1}{k_2 q_1}, \quad w^* = \frac{q_3^* k_1^*}{k_2^* q_1^*} \tag{9}$$

related to the reduced cross ratios of (3) by

$$|w|^2 = \frac{\tilde{u}_2}{\tilde{u}_3} = \frac{u_2}{u_3}, \quad w = |w| e^{i(\phi_2 - \phi_3)}, \quad \cos(\phi_2 - \phi_3) = \frac{1 - \tilde{u}_2 - \tilde{u}_3}{2\sqrt{\tilde{u}_2 \tilde{u}_3}} = \frac{1 - u_1 - u_2 - u_3}{2\sqrt{u_2 u_3}}. \tag{10}$$

The energy behavior of the remainder function is determined by the reggeon intercept

$$\omega(\nu, n) = -a E_{\nu,n}, \tag{11}$$

where a is the perturbation theory parameter

$$a = \frac{\alpha_s N_c}{2\pi} \tag{12}$$

and $E_{\nu,n}$ is the eigenvalue of the BFKL Kernel in the adjoint representation given by

$$E_{\nu,n} = -\frac{1}{2} \frac{|n|}{\nu^2 + \frac{n^2}{4}} + \psi\left(1 + i\nu + \frac{|n|}{2}\right) + \psi\left(1 - i\nu + \frac{|n|}{2}\right) - 2\psi(1). \tag{13}$$

Here $\psi(z) = \Gamma'(z)/\Gamma(z)$ and $\gamma = -\psi(1)$ is the Euler constant. The two loop LLA expression for remainder function in the BFKL approach was first found from (7) and (8) in ref. [7]

$$R^{(2)\,LLA}_{BFKL} \simeq \frac{i\pi}{2}\ln(1-u_1)\ln\tilde{u}_2\ln\tilde{u}_3 = \frac{i\pi}{2}\ln(1-u_1)\ln|1+w|^2\ln\left|1+\frac{1}{w}\right|^2. \tag{14}$$

This result was shown by Schabinger [22] to agree numerically with the expression obtained by analytic continuation of the remainder function found by Drummond, Henn, Korchemsky and Sokatchev [11] from Wilson Loop/Scattering Amplitude duality. The remainder function (14) was then explicitly confirmed by two of the authors [15] performing the analytic continuation of the Goncharov-Spradlin-Vergu-Volovich (GSVV) two-loop expression [14]. The analytic continuation allowed also to extract the next-to-leading contribution, not yet available from the BFKL approach

$$R^{(2)\,NLLA} \simeq \frac{i\pi}{2}\ln|w|^2\ln^2|1+w|^2 - \frac{i\pi}{3}\ln^3|1+w|^2 + i\pi\ln|w|^2\left(\mathrm{Li}_2(-w)+\mathrm{Li}_2(-w^*)\right)$$
$$-i2\pi\left(\mathrm{Li}_3(-w)+\mathrm{Li}_3(-w^*)\right). \tag{15}$$

The LLA term in (14) is pure imaginary and symmetric under $w \to 1/w$ transformation in accordance with (8). The next-to-leading (NLLA) contribution in (15) is also pure imaginary and has the same symmetry. Both of the contributions are pure imaginary due to a cancellation of the real part coming from the Mandelstam cut, Regge pole and a phase present in the BDS amplitude as was shown by one of the authors [17]. Starting at three loops this cancellation does not happen anymore and the real part gives a non-vanishing contribution at the next-to-leading level. The analysis of ref. [17] based on analyticity and other general properties of the scattering amplitudes resulted in a formulation of the dispersion-like relation for the real and imaginary parts of the remainder function in the Regge kinematics at the Mandelstam region

$$R\,e^{i\pi\,\delta} = \cos\pi\omega_{ab} + i\int_{-i\infty}^{i\infty}\frac{d\omega}{2\pi i}\,f(\omega)\,e^{-i\pi\omega}\,(1-u_1)^{-\omega}\,, \tag{16}$$

where the first term in RHS corresponds to the contribution of the Regge pole. This term as well as the phase δ in LHS of (16) are obtained directly from the BDS formula [17]

$$\delta = \frac{\gamma_K}{8}\,\ln(\tilde{u}_2\tilde{u}_3)\,,\ \omega_{ab} = \frac{\gamma_K}{8}\,\ln\frac{\tilde{u}_2}{\tilde{u}_3}. \tag{17}$$

The second terms in RHS of (16) stands for the contribution of the Mandelstam cut. The coefficient $\gamma_K \simeq 4a$ is the cusp anomalous dimension known to an arbitrary order of the perturbation theory. The only unknown piece in Eq. 16 is the real function $f(\omega)$, which contains the Mandelstam cut in ω and depends only on the transverse particle momenta and has no energy dependence. In the leading logarithmic approximation $f(\omega)$ can be extracted from (8) and reads

$$f^{LLA}(\omega) = \frac{a}{2}\sum_{n=-\infty}^{\infty}\int d\nu\frac{1}{\omega-\omega(\nu,n)}\frac{(-1)^n}{\nu^2+\frac{n^2}{4}}\,(w^*)^{i\nu-\frac{n}{2}}\,(w)^{i\nu+\frac{n}{2}}\,, \tag{18}$$

where $\omega(\nu,n)$ is defined in (11).

The dispersion-like relation in (16) was used for calculating the three loop contributions to $R^{(3)}_6$ (leading imaginary and the sub-leading real terms) in the multi-Regge kinematics

$$R^{(3)\,LLA}_{BFKL} = i\Delta^{(3)\,LLA}_{2\to4}/a^3 = i\pi\frac{1}{4}\ln^2(1-u_1)\Big(\ln|w|^2\ln^2|1+w|^2 - \frac{2}{3}\ln^3|1+w|^2 \tag{19}$$
$$-\frac{1}{4}\ln^2|w|^2\ln|1+w|^2 + \frac{1}{2}\ln|w|^2\left(\mathrm{Li}_2(-w)+\mathrm{Li}_2(-w^*)\right) - \mathrm{Li}_3(-w) - \mathrm{Li}_3(-w^*)\Big)$$

and

$$\Re\left(R^{(3)\,NLLA}_{BFKL}\right) = \frac{\pi^2}{4}\ln(1-u_1)\left(\ln|w|^2\ln^2|1+w|^2 - \frac{2}{3}\ln^3|1+w|^2\right. \tag{20}$$
$$\left. -\frac{1}{2}\ln^2|w|^2\ln|1+w|^2 - \ln|w|^2\left(\mathrm{Li}_2(-w)+\mathrm{Li}_2(-w^*)\right) + 2\mathrm{Li}_3(-w) + 2\mathrm{Li}_3(-w^*)\right).$$

As in the two loop case, both (19) and (20) are symmetric under $w \to 1/w$ transformation, which is obvious from (8) and corresponds to the target-projectile symmetry of the scattering amplitude. The corrections, subleading in the logarithm of the energy, are not captured by (8) and require some knowledge of the next-to-leading impact factor and the intercept of the BFKL eigenvalue in the adjoint representation. While the latter is still to be found from the next-to-leading BFKL equation, the correction to the impact factor was obtained in ref. [16] extracting it from (15). This result showed an intriguing relation between next-to-leading corrections to the impact factor at two loops and the three loop leading logarithmic contribution (see sections 4 and 5 of ref. [16] for more details).

In the next section we discuss the collinear limit of the scattering amplitudes in the multi-Regge kinematics, which is similar to the double logarithmic approximation with an overlapping of the BFKL and DGLAP approaches.

3 Collinear and Regge kinematics

In this section we consider the collinear limit of the amplitudes in the multi-Regge kinematics. In this limit two neighboring particles become collinear and one of the energy invariants tends to zero. Among a variety of possibilities we pick up one case, where the initial particle with momentum p_B in Fig. 1 is collinear to a particle in the final state with momentum $p_{B'}$. This corresponds to $t_3 \to 0$ and thus to $u_2 \to 0$. At this point we should take care about one fine point. Namely, in our analysis based on the BFKL approach we choose the largest scale dictated by the multi-Regge kinematics, which produces the leading logarithms in each order of the perturbation theory in the effective summation parameter, which is $a\ln(1-u_1)$. Taking the collinear limit we introduce another, a potentially larger parameter, which at the first sight does not satisfy the basic assumptions of the BFKL approach. In a general case the collinear and Regge limits do not necessarily commute, but having a physical intuition from the BFKL and DGLAP equations we have all reasons to believe that these two limits are interchangeable. This is indeed the case as will be shown later.

We start with taking $u_2 \to 0$ ($t_3 \to 0$) faster than $1-u_1$, in other words we assume that the reduced cross ratio $\tilde{u}_2$ in (3) vanishes in contrast to the multi-Regge kinematics in (2), where it is kept to be of the order of unity. It is also known that in the collinear kinematics $u_3 \simeq 1-u_1$ for $u_2 \to 0$ and we will use that fact later. So that now we choose the following kinematics in terms of the dual conformal cross ratios (compare to the Regge kinematics in (2))

$$1-u_1 \to +0, \;\; u_2 \to +0, \;\; u_3 \to +0, \;\; \frac{u_2}{1-u_1} = \tilde{u}_2 \to +0, \;\; \frac{u_3}{1-u_1} = \tilde{u}_3 \simeq 1, \tag{21}$$

which in terms of w and w^* implies (see (10))

$$1-u_1 \to +0, \;\; |w| \to +0, \;\; \cos(\phi_2-\phi_3) \simeq \mathcal{O}(1). \tag{22}$$

In the BFKL approach one sums large logarithms $\ln(1-u_1)$ keeping $|w|$ finite. We approach the limit (22) by taking $\ln|w|$ to be of the order (though not larger) of $\ln(1-u_1)$, which is still compatible with the BFKL resummation.

Expanding in this limit the two- and three-loop results for the remainder function in (14), (15), (19) and (20) we obtain

$$R^{(2)\,LLA}_{BFKL} + R^{(2)\,NLLA} \simeq -i2\pi \cos(\phi_2 - \phi_3)\,|w|\,\left(\ln(1-u_1)\ln|w| + 2\ln|w| - 2\right) \tag{23}$$

and

$$R^{(3)\,LLA}_{BFKL} \simeq -\frac{i\pi}{2}\ln^2(1-u_1)\cos(\phi_2-\phi_3)\,|w|\,\ln^2|w| \tag{24}$$

as well as

$$\Re\left(R^{(3)\,NLLA}_{BFKL}\right) \simeq -\pi^2\ln(1-u_1)\cos(\phi_2-\phi_3)\,|w|\,\ln^2|w|. \tag{25}$$

The BFKL approach allows to calculate in the double logarithmic limit the leading imaginary and real contributions to any order of the perturbation theory. In the collinear limit $|w|\to 0$ the largest contribution in the integral over ν in (8) comes from the poles at $\nu = -in/2$ for $n=1$. The details of this calculation are presented in appendix A and the result is expressed in terms of the modified Bessel functions $I_k(z)$ as follows. The contribution leading in both $\ln(1-u_1)$ and $\ln|w|$ reads

$$R^{DLLA}_{BFKL} \simeq i2\pi a\cos(\phi_2-\phi_3)\,|w|\left(1 - I_0\left(2\sqrt{a\ln|w|\ln(1-u_1)}\right)\right), \tag{26}$$

while the real part of the contribution suppressed in the logarithm of the energy $\ln(1-u_1)$ (NDLLA term) is given by

$$\begin{aligned}\Re\left(R^{NDLLA}_{BFKL}\right) &\simeq 2\pi^2 a^{3/2}\cos(\phi_2-\phi_3)\,|w|\,\ln|w|\frac{I_1\left(2\sqrt{a\ln|w|\ln(1-u_1)}\right)}{\ln(1-u_1)} \\ &+4\pi^2 a\cos(\phi_2-\phi_3)\,|w|\,\ln|w|\left(1 - I_0\left(2\sqrt{a\ln|w|\ln(1-u_1)}\right)\right) - 2\pi^2a^2\cos(\phi_2-\phi_3)|w|\ln|w|.\end{aligned} \tag{27}$$

The collinear limit of the remainder function for the six-gluon planar MHV amplitude was earlier considered by Alday, Gaiotto, Maldacena, Sever and Vieira (AGMSV) in ref. [19]. They suggested that introducing the following parametrization of the dual conformal cross ratios

$$u_2 = \frac{1}{\cosh^2\tau},\quad u_1 = \frac{e^{\sigma}\sinh\tau\tanh\tau}{2(\cos\phi+\cosh\tau\cosh\sigma)},\quad u_3 = \frac{e^{-\sigma}\sinh\tau\tanh\tau}{2(\cos\phi+\cosh\tau\cosh\sigma)} \tag{28}$$

one can write the remainder function in the collinear limit $\tau\to\infty$ in a rather compact way

$$R^{(\ell)}_{OPE} \sim \cos\phi\, e^{-\tau}\frac{(-1)^{\ell-1}\tau^{\ell-1}}{(\ell-1)!}\int dp\, e^{ip\sigma}c^0(p)\gamma_1^{\ell-1}(p), \tag{29}$$

where ℓ is a number of loops and $\gamma_1(p)$ is the anomalous dimension of high spin operators considered in the operator product expansion of ref. [19] (see also a paper of Basso [23])

$$\gamma_1(p) = \psi\left(\frac{3}{2}+\frac{ip}{2}\right) + \psi\left(\frac{3}{2}-\frac{ip}{2}\right) - 2\psi(1). \tag{30}$$

The function $c^0(p)$ can found from one loop (i.e. the BDS expression) and reads

$$c^{(0)}(p) \propto \frac{1}{1+p^2}\frac{\pi}{\cosh\frac{\pi p}{2}}. \tag{31}$$

Indeed, for the $2 \to 4$ amplitude we can write the BDS at one loop up to irrelevant terms that depend on μ^2 and ϵ as

$$I_6 + F_6 \simeq \frac{1}{2}\ln s \ln s_2 - \frac{1}{2}\ln s \ln t_1 - \frac{1}{2}\ln s \ln t_3 - \frac{1}{2}\ln s_1 \ln s_2 - \frac{1}{2}\ln s_1 \ln t_1 \tag{32}$$
$$+\frac{1}{2}\ln s_1 \ln t_3 - \frac{1}{2}\ln s_2 \ln s_3 + \frac{1}{2}\ln s_3 \ln t_1 - \frac{1}{2}\ln s_3 \ln t_3 + \frac{3\pi^2}{4} + R_1,$$

where $R^{(1)}$ is a function of only anharmonic ratios u_i

$$R^{(1)} = -\frac{1}{2}\sum_{i=1}^{3}\left(\frac{1}{2}\ln^2 u_i + \mathrm{Li}_2(1-u_i)\right). \tag{33}$$

In the collinear limit $\tau \to \infty$ we obtain

$$R^{(1)} \simeq -\tau^2 + 2\tau \ln 2 - \frac{\pi^2}{6} - \ln^2 2 - \sigma^2 + \cos\phi\, e^{-\tau} h_0(\sigma), \tag{34}$$

where

$$h_0(\sigma) = \int_{-\infty}^{\infty} c^{(0)}(p) e^{ip\sigma} dp \tag{35}$$

for

$$c^{(0)}(p) = \frac{2}{1+p^2}\frac{1}{\cosh\frac{\pi p}{2}}. \tag{36}$$

This simple one-loop analysis allows us to fix the normalization in (31). However, taking into account some ambiguity in expressing the finite part of the BDS formula in terms of the anharmonic ratios we fix the normalization of $c^{(0)}(p)$ using the two-loop remainder function. Both of them give the same expression for $c^{(0)}(p)$ as we show later. It is worth emphasizing that the one-loop BDS “remainder function“ $R^{(1)}$ in (33) can be compactly written as

$$R^{(1)} = \frac{1}{2}\sum_{i=1}^{3}\mathrm{Li}_2\left(1 - \frac{1}{u_i}\right). \tag{37}$$

Polylogarithmic functions of the same argument appear also in the two-loop remainder function suggesting an intimate relation between the BDS amplitude and its corrections.

The expression in (29) was also shown to agree numerically [19] at two loops with the result of Goncharov, Spradlin, Vergu and Volovich (GSVV) [14]. The GSVV remainder function was derived for quasi-multi Regge kinematics [12, 13], but it was argued to be valid also in a general kinematics for positive values of the dual conformal cross ratios. The analytic continuation in one of the dual conformal cross ratios, namely the one given by (5), with subsequent multi-Regge limit of (2) reproduces the LLA BFKL result [15] for the physical Mandelstam region. Naturally, an important question to be asked is whether or not one can perform a similar analytic continuation of the AGMSV expression in (29) to find an agreement or disagreement with the BFKL analysis. In the attempt of answering this question we immediately face a difficulty of treating the cosine factor in (29). In deriving the remainder function in the collinear limit (29) it was assumed in ref.[19] that the absolute value of $\cos\phi$ is finite and is much smaller than τ. It is indeed the case also in the collinear and Regge kinematics we are interested in (see (21)). However, it is easy to see from the definition

$$\cos\phi = \frac{u_1 + u_2 + u_3 - 1}{2\sqrt{u_1 u_2 u_3}}, \tag{38}$$

that in the course of the analytic continuation (5) at $u_1 = |u_1|e^{-i\pi}$ the numerator becomes of the order of 2, while the denominator is still small and thus (38) is not limited anymore. This means that one cannot directly apply the analytic continuation (5) to the AGMSV expression, because the unlimited growth of $\cos\phi$ at $u_1 = |u_1|e^{-i\pi}$ does not satisfy the assumptions of the collinear expansion and therefore (29) is not always valid during the analytic continuation. We face a similar problem performing the analytic continuation (5) of the GSVV remainder function, when the value of $1 - u_1$ at $u_1 = |u_1|e^{-i\pi}$ becomes of the order of 2 and thus does not satisfy the first condition in the multi-Regge kinematics given by (2). However, the GSVV expression is valid for all positive values (arbitrary kinematics) of the dual conformal cross ratios and therefore its continuation does not lead to any difficulty.

In the case of the collinear expansion the condition of having $\cos\phi$ being limited along the path of the analytic continuation forces us to modify the simple circular path for u_1 in (5) and/or change also paths of u_2 and u_3, which are trivial in (5). The initial and the final points of the analytic continuation should be the same, and a deformation of the continuation path is possible under condition that we do not cross any singularities of the remainder function.

It is plausible that one can deform the path of the analytic continuation of the remainder function in such a way that $\cos\phi$ in (29) remains limited along a new path. In other words the new path could be compatible with the collinear kinematics. To prove it in the case of the two-loop GSVV expression one should consider the analytic continuation of the function of two variables u_1 and u_3, keeping u_2 fixed and small. We hope to do this in the future. Below we assume that such a deformation of the path of the analytic continuation does exist.

Note that in general kinematics we defined (see (28))

$$\frac{u_1}{u_3} = e^{2\sigma}, \quad \sigma = \frac{1}{2}\ln\frac{u_1}{u_3} \tag{39}$$

and thus the analytic continuation of (29) along a path given by (5) in the complex σ-plane would mean a simple shift

$$\sigma \Rightarrow \sigma - i\pi, \tag{40}$$

where σ is large and positive for multi-Regge kinematics given in (2). We name this path in the σ-space *the path* **A** and argue that this continuation is not valid for expressions, where the collinear limit was performed first.

On the other hand in the collinear kinematics (provided $\cos\phi$ is of the order of unity) the cross ratios can be approximated by (see (28))

$$u_2 \simeq 4e^{-2\tau}, \quad u_1 \simeq \frac{e^{\sigma}}{2\cosh\sigma}, \quad u_3 \simeq \frac{e^{-\sigma}}{2\cosh\sigma} \tag{41}$$

with a simple relation $u_3 \simeq 1 - u_1$. So that we can plug this relation in (39) and redefine

$$\sigma \simeq \frac{1}{2}\ln\frac{u_1}{1-u_1}, \tag{42}$$

which gives the same expression for the function of σ in (29) in the collinear limit, but changes the path of the analytic continuation in the σ-plane for $u_1 = |u_1|e^{-i2\pi}$. In fact, the use of (42) instead of (39) in the expression (29) means the redefinition of this expression in the region beyond the collinear limit. We believe, that in the case, when σ depends only on u_1, it is possible to prove, that the path of the analytic continuation can be deformed in such a way that $\cos\phi$ remains restricted along this path and, as a result, (29) can be used for the continuation.

We name the path corresponding to our new definition of σ in (42) *the path* **B**, and assume that the analytic continuation with this deformed path is valid also for expression where the collinear limit was performed first. In particular, we see that in contrast to the analytic continuation along the path **A** the cosine factor can be made finite at the point $u_1 = |u_1|e^{-i\pi}$ since

the numerator can have the same smallness as the denominator provided u_3 is adjusted in the corresponding way. Note that this analytic continuation is different from one applied to the GSVV remainder function in ref. [15], because u_3 is not kept fixed anymore and acquires some phase. However, in the course of the analytic continuation along the path **B** we never cross the imaginary axis in the complex u_3-plane, i.e. never go to the negative real values of u_3 and thus do not cross the singularities of the GSVV expression. The paths **A** and **B** in the u_3-space are illustrated in Fig. 3, where ψ is defined by $u_1 = |u_1|e^{-i\psi}$ and changes from 0 to 2π in the course of the analytic continuation.

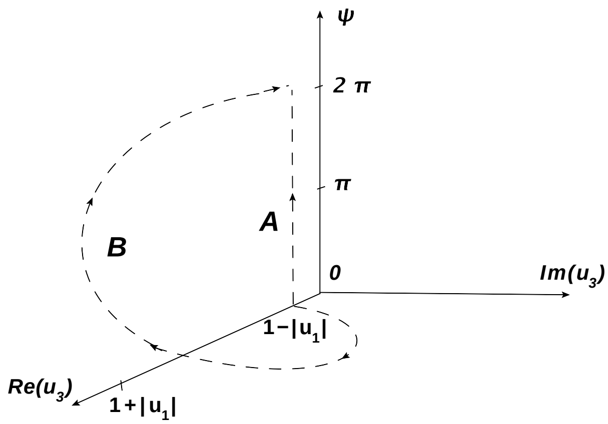

Figure 3: The paths **A** and **B** in the u_3-space. The phase ψ is defined by $u_1 = |u_1|e^{-i\psi}$ and takes values between 0 and 2π in the course of the analytic continuation. The value of u_3 is the same at the initial and final points of the paths **A** and **B**.

The fact that u_3 never crosses the imaginary axis allows us to deform smoothly the path of the continuation and so that it could be compatible with the collinear kinematics. To demonstrate this fact, we performed the analytic continuation (5) of the GSVV remainder function for arbitrary, but small values of u_2. Then we took the collinear limit $u_2 \to 0$ of the continued function, substituted $u_3 \simeq 1 - u_1$ for an arbitrary value of u_1 (i.e. no Regge limit) reproducing the result of the analytic continuation of (29) along the path **B** for arbitrary positive σ. This way we show that the analytic continuation of AGMSV expression in (29) along path **B** is justified.

As the next step in our analysis we want to apply the analytic continuation **B** to the AGMSV expression in (29) at higher loops and after taking the $\sigma \to +\infty$ limit compare the result to the one obtained in the BFKL approach. This requires a knowledge of the overall constant, which is possible to fix at two loops expanding the GSVV remainder function at $\tau \to \infty$. We find that it can be written as

$$R^{(2)}_{GSVV} \simeq \cos\phi\, e^{-\tau}\left(-\tau h_1(\sigma) + h_1^{sub}(\sigma)\right) + \mathcal{O}\left(e^{-2\tau}\right), \tag{43}$$

where $h_1(\sigma)$ is a function calculated in ref. [19]

$$h_1(\sigma) = -2\cosh\sigma\left(2\ln\left(1+e^{2\sigma}\right)\ln\left(1+e^{-2\sigma}\right) - 4\ln\left(2\cosh\sigma\right)\right) - 8\sigma\sinh\sigma. \tag{44}$$

The sub-leading in τ contribution we extract from the GSVV expression

$$h_1^{sub}(\sigma) = -\frac{2}{3}\pi^2\sigma\cosh\sigma - 4\sigma^2\cosh\sigma - \frac{8}{3}\sigma^3\cosh\sigma + 4\sigma^2\cosh\sigma\ln 2 \tag{45}$$
$$-8\cosh\sigma\ln(2\cosh\sigma) + \frac{2}{3}\pi^2\cosh\sigma\ln(2\cosh\sigma) + 4\sigma^2\cosh\sigma\ln(2\cosh\sigma)$$
$$+8\cosh\sigma\ln(2\cosh\sigma)\ln 2 + 4\cosh\sigma\ln^2(2\cosh\sigma) - 4\cosh\sigma\ln^2(2\cosh\sigma)\ln 2$$
$$-\frac{4}{3}\cosh\sigma\ln^3(2\cosh\sigma) + 4\cosh\sigma\mathrm{Li}_3\left(-e^{-2\sigma}\right) - 8\sigma\sinh\sigma - 8\sigma\sinh\sigma\ln 2.$$

The functions $h_1(\sigma)$ and $h_1^{sub}(\sigma)$ are symmetric in $\sigma \to -\sigma$ and vanish at $\sigma \to \pm\infty$. The explicit expression for $h_1(\sigma)$ allows to fix the overall coefficient of the AGMSV remainder function in (29). The normalization fixed using two-loop GSVV expression coincides with the normalization we fixed using only BDS one-loop expression (see (36) and the text whereafter). Thus we can write the AGMSV remainder function (29) in the exponential form

$$R_{OPE} \simeq a\cos\phi\, e^{-\tau}\int_{-\infty}^{\infty} c^0(p)\left(e^{-a\tau\gamma_1(p)} - 1\right)e^{ip\sigma}dp \simeq a^2 R_{OPE}^{(2)} + a^3 R_{OPE}^{(3)} + ... \tag{46}$$

with

$$c^{(0)}(p) = \frac{2}{1+p^2}\frac{1}{\cosh\frac{\pi p}{2}}. \tag{47}$$

Introducing

$$h_k(\sigma) = \int_{-\infty}^{\infty} c^{(0)}(p)\,\gamma_1^k(p)\,e^{ip\sigma}dp \tag{48}$$

the AGMSV remainder function (46) can be written as

$$R_{OPE} \simeq a\cos\phi\, e^{-\tau}\sum_{k=1}^{\infty}\frac{(-a\tau)^k}{k!}h_k(\sigma). \tag{49}$$

The knowledge of $h_1^{sub}(\sigma)$ gives a possibility of calculating the one-loop correction to the "coefficient function" $c^0(p)$. To find the next-to-leading correction to the remainder function in the collinear limit one needs also corrections to the anomalous dimensions $\gamma_1(p)$ in (30).

Next we investigate the analytic structure of the AGMSV remainder function in the complex σ-plane. Both $h_1(\sigma)$ and $h_1^{sub}(\sigma)$ have the same branch cuts in the complex σ-plane starting at $\pm i\pi(2n+1)/2$ for $n = 0, 1, 2, ...$ as illustrated in Fig. 4. From Fig. 4 one can see the difference between the two paths **A** and **B** of the analytic continuation. The path **A** is linear in the σ-plane and does not cross horizontal branch cuts, while the path **B** is non-trivial and does cross the branch cut. For the purpose of the present discussion the main difference between the two cases is the fact that after the analytic continuation along the path **A** functions $h_k(\sigma)$ and $h_1^{sub}(\sigma)$ vanish for $\sigma \to +\infty$ (Regge limit), while for the path **B** they give a non-vanishing term compatible with the logarithmic contributions in the BFKL approach. For the details of the analytic continuation the reader is referred to the appendix C and here we present the main results.

We found that the analytic continuation along the path **B** together with the subsequent Regge limit $\sigma \to +\infty$ of the AGMSV remainder function (43) fully reproduce the BFKL result at two loops for the double (collinear and Regge) logarithmic limit of the remainder function given by (23). In particular, we establish a connection between the conformal spin defined in ref. [19] and the conformal spin used in the BFKL approach noting that in the multi-Regge kinematics (see (10) and (38))

$$\cos\phi \simeq -\cos(\phi_2 - \phi_3), \tag{50}$$

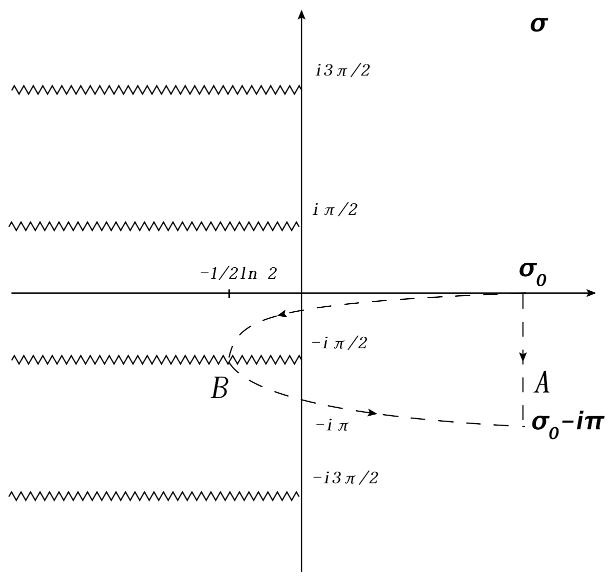

Figure 4: The cut structure of $h_k(\sigma)$ and $h_1^{sub}(\sigma)$. The figure illustrates the paths **A** and **B** of the analytic continuation. σ_0 denotes some starting point of the analytic continuation. Both of the paths have the same starting and final points. The Regge kinematics corresponds to $\sigma_0 \to +\infty$.

which after the analytic continuation becomes

$$\cos\phi \Rightarrow -\cos\phi \simeq \cos(\phi_2 - \phi_3). \tag{51}$$

To justify our guess for the analytic continuation along the path **B** we perform the well-grounded analytic continuation (5) for a general kinematics (used in confirming the BFKL result in the multi-Regge kinematics) of the Goncharov-Spradlin-Vergu-Volovich (GSVV) remainder function. Then we expand the continued GSVV expression for $\tau \to \infty$ and find that it coincides with the result of the analytic continuation along the path **B** of the AGMSV remainder function for a positive fixed value of the parameter σ (i.e. not necessarily in the Regge kinematics, where $\sigma \to +\infty$). This confirms the validity of the continuation along the path **B** as well as it shows commutativity of the collinear and Regge limits. One can expect the commutativity of these two limits from comparison of the BFKL (Regge kinematics) and DGLAP (Bjorken kinematics) equations, which coincide in the double logarithmic limit.

Using the definition of $h_k(\sigma)$ in (23) we calculate analytically the AGMSV remainder function at three loops $R^{(3)}_{OPE}$ in (46) (see appendix B for more details)

$$\begin{aligned} h_2(\sigma) &= \int_{-\infty}^{\infty} c^0(p)\gamma_1^2(p)e^{ip\sigma}dp = -\frac{\pi^2}{3}e^{-\sigma} - 4e^{-\sigma}\sigma^2 - \frac{2}{3}\pi^2\sigma\cosh\sigma \\ &+16\sigma^2\cosh\sigma + \frac{8}{3}\sigma^3\cosh\sigma + 24\cosh\sigma\ln(2\cosh\sigma) + \frac{2}{3}\pi^2\cosh\sigma\ln(2\cosh\sigma) \\ &-8\sigma^2\cosh\sigma\ln(2\cosh\sigma) - 16\cosh\sigma\ln^2(2\cosh\sigma) + \frac{16}{3}\cosh\sigma\ln^3(2\cosh\sigma) \\ &+8\sigma\cosh\sigma\,\mathrm{Li}_2(-e^{-2\sigma}) + 8\cosh\sigma\,\mathrm{Li}_3(-e^{-2\sigma}) - 24\sigma\sinh\sigma + 4\sinh\sigma\,\mathrm{Li}_2(-e^{-2\sigma}). \end{aligned} \tag{52}$$

The analytic continuation along the path **B** of $R^{(3)}_{OPE}$ reproduces the BFKL remainder function in the double logarithmic approximation at three loops given by (24) and (25), namely

$$\begin{aligned} R^{(3)}_{OPE} \overset{\text{path}\mathbf{B}}{\Longrightarrow} &-\frac{i\pi}{2}\ln^2(1-u_1)\cos(\phi_2-\phi_3)\;|w|\;\ln^2|w| - \pi^2\ln(1-u_1)\cos(\phi_2-\phi_3)\;|w|\;\ln^2|w| \\ &-i3\pi\ln(1-u_1)\cos(\phi_2-\phi_3)\;|w|\;\ln^2|w|. \end{aligned} \tag{53}$$

The first two terms in RHS of (53) coincide with the corresponding BFKL expressions in (24) and (25), while the last term is currently not accessible in the BFKL analysis and brings some new information about the next-to-leading eigenvalue of the BFKL eigenvalue in the adjoint representation. For higher loops we need an analytic form of $h_k(\sigma)$, which are not considered here due to the complexity of the calculations for $k > 2$.

The analytic continuation along the path **B** for the AGMSV remainder function is technically more involved than a simple shift $\sigma \Rightarrow \sigma - i\pi$ in the path **A**. Despite the fact that the continuation along the path **A** is not applicable for the AGMSV remainder function as we discussed earlier, we still can make a use of it at higher loops due to an interesting transformation property of $h_k(\sigma)$

$$\begin{array}{ll} \textit{continuation along the path } \mathbf{A}: & h_k(\sigma) \Rightarrow -h_k(\sigma) + \Delta_k^*(-\sigma) \\ \textit{continuation along the path } \mathbf{B}: & h_k(\sigma) \Rightarrow -h_k(\sigma) + \Delta_k(\sigma), \end{array} \tag{54}$$

where $\Delta_k(\sigma)$ is some function of σ calculated in the appendix C for $k = 1$ and $k = 2$, which corresponds to two and three loops of the AGMSV remainder function. We checked that analytic continuation along the path **B** of $h_k(\sigma)$ can be obtained by complex conjugation and reversing the argument of the analytic continuation along the path **A** and vice versa at two and three loops. We believe that this property holds at higher loops as well. It is technically much easier to perform the analytic continuation with the path **A** and then calculate from it the required function $\Delta_k(\sigma)$ using the property (54).

The minus sign of $h_k(\sigma)$ that appears on RHS of (54) is related to the fact that after the analytic continuation $\cos\phi$ also changes the sign

$$\cos\phi \Rightarrow -\cos\phi \tag{55}$$

so that the AGMSV remainder function (46), which is a product of $h_k(\sigma)$ and $\cos\phi$, does not change the sign and only gets an additive discontinuity $-\cos\phi\,\Delta_k(\sigma)$ as expected.

We also made another intriguing observation, namely that the function $\Delta_k(\sigma)$ is much simpler than $h_k(\sigma)$ and can be obtained from the expression (48) with omitted $\cosh(\pi p/2)$ in the denominator

$$F_k(\sigma) = \int_{-\infty}^{\infty} c^{(0)}(p)\,\gamma_1^k(p)\left(2\cosh\frac{\pi p}{2}\right)\,e^{ip\sigma}dp = \int_{-\infty}^{\infty}\frac{4}{1+p^2}\gamma_1^k(p)\,e^{ip\sigma}dp \tag{56}$$

by changing the sign of σ and shifting it by $i\pi/2$

$$\Delta_k(\sigma) = F_k\left(-\sigma + \frac{i\pi}{2}\right). \tag{57}$$

We have checked this property at two and three loops, i.e. for

$$\begin{aligned} F_1(\sigma) &= 8e^{-\sigma}\pi + 8\pi\ln(1-e^{-2\sigma})\sinh\sigma, \\ F_2(\sigma) &= 24e^{-\sigma}\pi - 8e^{-\sigma}\pi\sigma + 32\pi\ln\left(1-e^{-2\sigma}\right)\sinh\sigma - 16\pi\sigma\ln\left(1-e^{-2\sigma}\right)\sinh\sigma \\ &\quad -16\pi\ln^2\left(1-e^{-2\sigma}\right)\sinh\sigma - 8\pi\ \mathrm{Li}_2\left(-e^{-2\sigma}\right)\sinh\sigma. \end{aligned} \tag{58}$$

The main advantage of this observation is that $F_k(\sigma)$ is much easier to calculate than the initial function $h_k(\sigma)$. We believe that the two properties (54) and (57) are intimately related to each other and their possible interpretation in terms of the energy discontinuities is presented in section 4.

3.1 Four and five loops

In the previous part of the paper we performed the analytic continuation of the AGMSV remainder function (46) at two and three loops, and then taking the Regge limit we reproduced the BFKL result (26) and (27) in the Double Leading Logarithmic Approximation (DLLA). In doing this we needed an explicit analytic form of the function $h_k(\sigma)$. Going beyond three loops (i.e. $h_2(\sigma)$) in (46) presents a technical challenge and we found it much easier to calculate contributions only of powers of $\gamma_1^{\pm}(p)$ defined by

$$\gamma_1(p) = \gamma_1^+(p) + \gamma_1^-(p), \quad \gamma_1^{\pm}(p) = \psi\left(\frac{3}{2} \pm \frac{ip}{2}\right) - \psi(1). \tag{59}$$

A possible physical interpretation of the functions $\gamma_1^{\pm}(p)$ is discussed in the next section. It turns out that the main contribution to the AGMSV remainder function in DLLA in the Mandelstam channel comes from the maximal powers of $\gamma_1^-(p)$ in (46). Each power of $\gamma_1^+(p)$ introduced a suppression by one power of $\ln(1-u_1)$ in terms, which are leading in $\ln|w|$. Generally, we consider the multi-loop contribution from $\gamma_1^+(p)$ and $\gamma_1^-(p)$ in separate, introducing

$$h_k^{+,\dots,-}(\sigma) = \int_{-\infty}^{\infty} c^0(p)\left(\gamma_1^+(p)\right)^m \left(\gamma_1^-(p)\right)^{k-m} e^{ip\sigma}dp, \tag{60}$$

where m is the number of powers of $\gamma_1^+(p)$ and $k = \ell - 1$ is related to a number of loops ℓ. The functions $h_k^{\pm}(\sigma)$ are calculated in appendix D and given for the two-loop case by

$$h_1^-(\sigma) = \int_{-\infty}^{\infty} c^0(p)\gamma_1^-(p)e^{ip\sigma}dp = 4e^{-\sigma}\sigma - \frac{1}{3}\pi^2 e^{-\sigma} + 4\cosh\sigma \ln\left(1+e^{-2\sigma}\right) \tag{61}$$
$$-2\cosh\sigma\ln^2\left(1+e^{-2\sigma}\right) - 4\cosh\sigma \mathrm{Li}_2\left(-e^{-2\sigma}\right)$$

and

$$h_1^+(\sigma) = h_1(\sigma) - h_1^-(\sigma) = 4\cosh\sigma\mathrm{Li}_2\left(-e^{-2\sigma}\right) + 6\sigma^2\cosh\sigma + 4e^{-\sigma}\sigma + \frac{1}{3}\pi^2 e^{-\sigma} \tag{62}$$
$$-4\sigma\cosh\sigma - 2\cosh\sigma\ln^2(2\cosh\sigma) - 4\sigma\cosh\sigma\ln(2\cosh\sigma) + 4\cosh\sigma\ln(2\cosh\sigma).$$

Note that it follows from the definitions (59) and (60) that they are related by $h_1^+(\sigma) = h_1^-(-\sigma)$. Similarly to the second line of (54), these functions after analytic continuation along path **B** also can be written as

$$h_k^{\pm}(\sigma) \Rightarrow -h_k^{\pm}(\sigma) + \Delta_k^{\pm}(\sigma), \tag{63}$$

where $\Delta_k^{\pm}(\sigma)$ read

$$\Delta_1^+(\sigma) = -4i\pi e^{\sigma}, \quad \Delta_1^-(\sigma) = -4i\pi e^{\sigma} + 8i\pi\sigma\cosh\sigma + 8i\pi\cosh\sigma\ln(2\cosh\sigma). \tag{64}$$

In the multi-Regge kinematics $\sigma \to \infty$ we get their respective contributions to the remainder function

$$R_{OPE}^{(2)+} = -\cos\phi e^{-\tau}\tau h_1^+(\sigma) \Longrightarrow -i2\pi\cos(\phi_2-\phi_3)|w|\ln|w| \tag{65}$$

and

$$R_{OPE}^{(2)-} = -\cos\phi\, e^{-\tau}\, \tau h_1^-(\sigma) \Longrightarrow -i2\pi\cos(\phi_2-\phi_3)|w|\ln|w|\ln(1-u_1) \tag{66}$$
$$-i2\pi\cos(\phi_2-\phi_3)|w|\ln|w|.$$

In (65) and (66) we omit terms not enhanced by $\ln|w|$, which are irrelevant for the present discussion. The full form of $R_{OPE}^{(2)\pm}$ is presented in appendix D. From (65) and (66) we see that

$R^{(2)+}_{OPE}$ is suppressed by one power of $\ln(1-u_1)$ with respect to $R^{(2)-}_{OPE}$. The $R^{(2)+}_{OPE}$ contribution is subleading and not captured by the double-logarithmic BFKL analysis. In order to find a corresponding contribution in the BFKL approach one needs to calculate the next-to-leading eigenvalue of the BFKL Kernel in the adjoint representation, and it is not currently available.

At the three loop level we also observe a similar situation, where the remainder function (46) is given by

$$R^{(3)}_{OPE} = R^{(3)++}_{OPE} + R^{(3)+-}_{OPE} + R^{(3)--}_{OPE} \tag{67}$$

with

$$R^{(3)++}_{OPE} = \cos\phi\, e^{-\tau}\frac{\tau^2}{2}h_2^{++}(\sigma),\;\; R^{(3)+-}_{OPE} = \cos\phi\, e^{-\tau}\tau^2 h_2^{+-}(\sigma),\;\; R^{(3)--}_{OPE} = \cos\phi\, e^{-\tau}\frac{\tau^2}{2}h_2^{--}(\sigma). \tag{68}$$

In the Double Leading Logarithmic Approximation for the Mandelstam channel we obtain

$$R^{(3)++}_{OPE} \Longrightarrow -i\pi|w|\ln^2|w| \tag{69}$$

and

$$\begin{aligned} R^{(3)+-}_{OPE} &\Longrightarrow -i2\pi\cos(\phi_2-\phi_3)|w|\ln(1-u_1)\ln^2|w| + \frac{i\pi^3}{3}\cos(\phi_2-\phi_3)|w|\ln^2|w| \\ &-i4\pi\cos(\phi_2-\phi_3)|w|\ln^2|w| \end{aligned} \tag{70}$$

as well as

$$\begin{aligned} R^{(3)--}_{OPE} &\Longrightarrow -\frac{i\pi}{2}\cos(\phi_2-\phi_3)|w|\ln^2|w|\ln^2(1-u_1) \\ &-\pi^2\cos(\phi_2-\phi_3)|w|\ln^2|w|\ln(1-u_1). \end{aligned} \tag{71}$$

It is clear from (69), (70) and (71) that each power of $\gamma_1^+(p)$ brings an additional suppression by one power of $\ln(1-u_1)$. We expect this to happen also at higher loops and argue that the main contribution in DLLA in the Mandelstam channel comes from the maximal power of $\gamma_1^-(p)$ in (60), namely for $m=0$

$$h_k^{-,\ldots,-}(\sigma) = \int_{-\infty}^{\infty} c^0(p)\left(\gamma_1^-(p)\right)^k e^{ip\sigma}dp, \tag{72}$$

which we call $h_k^-(\sigma)$ for short. The functions $h_3^-(\sigma)$ and $h_4^-(\sigma)$, which corresponds to 4 and 5 loops respectively, were calculated in appendix D. The function $h_3^-(\sigma)$ is given by

$$\begin{aligned} h_3^-(\sigma) &= \int_{-\infty}^{\infty} c^0(p)\left(\gamma_1^-(p)\right)^3 e^{ip\sigma}dp = -\pi^2 e^{-\sigma} + 4\sigma e^{-\sigma} + \frac{4}{15}\pi^4\cosh\sigma - e^{\sigma}\pi^2\ln\left(1+e^{-2\sigma}\right) \\ &+4\cosh\sigma\ln\left(1+e^{-2\sigma}\right) - 6e^{-\sigma}\sigma\ln^2\left(1+e^{-2\sigma}\right) - 6\cosh\sigma\ln^2\left(1+e^{-2\sigma}\right) + 12\sigma\cosh\sigma\ln^2\left(1+e^{-2\sigma}\right) \\ &+2e^{-\sigma}\ln^3\left(1+e^{-2\sigma}\right) + 3e^{\sigma}\ln^3\left(1+e^{-2\sigma}\right) + 4\sigma\cosh\sigma\ln^3\left(1+e^{-2\sigma}\right) - 6e^{-\sigma}\mathrm{Li}_2\left(-e^{-2\sigma}\right) \\ &-2e^{\sigma}\mathrm{Li}_2\left(-e^{-2\sigma}\right) + 6e^{-\sigma}\ln\left(1+e^{-2\sigma}\right)\mathrm{Li}_2\left(-e^{-2\sigma}\right) - 6\cosh\sigma\ln^2\left(1+e^{-2\sigma}\right)\mathrm{Li}_2\left(-e^{-2\sigma}\right) \\ &-4\cosh\sigma\mathrm{Li}_2\left(-e^{-2\sigma}\right)^2 - 6e^{-\sigma}\mathrm{Li}_3\left(-e^{-2\sigma}\right) - 2e^{\sigma}\mathrm{Li}_3\left(-e^{-2\sigma}\right) + 4\cosh\sigma\ln\left(1+e^{-2\sigma}\right)\mathrm{Li}_3\left(-e^{-2\sigma}\right) \\ &-6e^{\sigma}\mathrm{Li}_3\left(\frac{1}{1+e^{-2\sigma}}\right) - 12\cosh\sigma\ln\left(1+e^{-2\sigma}\right)\mathrm{Li}_3\left(\frac{1}{1+e^{-2\sigma}}\right) - 4\cosh\sigma\mathrm{Li}_4\left(-e^{-2\sigma}\right) \\ &-24\cosh\sigma\mathrm{Li}_4\left(\frac{1}{1+e^{-2\sigma}}\right) + 12\cosh\sigma\mathrm{Li}_{2,2}\left(-e^{-2\sigma}\right) + 6e^{\sigma}\zeta_3 - 12\cosh\sigma\ln\left(1+e^{-2\sigma}\right)\zeta_3, \end{aligned} \tag{73}$$

while the expression for $h_4^-(\sigma)$ is rather lengthy and is presented in appendix D (see (D.24)) in terms of the harmonic polylogarithms (HPL) [24].

The analytic continuation of $h_3^-(\sigma)$ and $h_4^-(\sigma)$ along path **B** with the subsequent Regge limit allows us to find the OPE remainder function (46) in the Mandelstam region in the Double Leading Logarithmic Approximation (DLLA). At four and five loops they read

$$R_{OPE}^{(4)-} = -\cos\phi\, e^{-\tau}\,\frac{\tau^3}{3!}h_3^-(\sigma) \Longrightarrow -\frac{i\pi}{18}\cos(\phi_2-\phi_3)|w|\ln^3|w|\ln^3(1-u_1) \tag{74}$$
$$-\frac{\pi^2}{3}\cos(\phi_2-\phi_3)|w|\ln^3|w|\ln^2(1-u_1) - \frac{i\pi}{6}\cos(\phi_2-\phi_3)|w|\ln^3|w|\ln^2(1-u_1)$$

and

$$R_{OPE}^{(5)-} = \cos\phi\, e^{-\tau}\,\frac{\tau^4}{4!}h_4^-(\sigma) \Longrightarrow -\frac{i\pi}{288}\cos(\phi_2-\phi_3)|w|\ln^4|w|\ln^4(1-u_1) \tag{75}$$
$$-\frac{\pi^2}{24}\cos(\phi_2-\phi_3)|w|\ln^4|w|\ln^3(1-u_1) - \frac{i\pi}{72}\cos(\phi_2-\phi_3)|w|\ln^4|w|\ln^3(1-u_1).$$

The first two terms in RHS of the remainder functions $R_{OPE}^{(4)-}$ and $R_{OPE}^{(5)-}$ reproduce the BFKL result in (26) and (27), while the last term is beyond the applicability of the double-logarithmic BFKL analysis and requires a knowledge of the NLO impact factor, calculated by of the authors in ref. [16] as well as the corrections to the eigenvalue of the BFKL Kernel in the adjoint representation. This can be obtained from the NLO BFKL Kernel in the adjoint representation found by Fadin and Fiore [25, 26].

In this section we showed that in order to reproduce known BFKL results in the double-logarithmic approximation up to five-loop level, it is enough to consider only a part of the anomalous dimension in (46). Namely, all of the leading terms come from $\gamma_1^-(p)$ in (59), while each power of $\gamma_1^+(p)$ introduces a suppression in one power of $\ln(1-u_1)$ in the Mandelstam region. In the next section we discuss this observation and argue that it could be a sign for a non-multiplicative renormalization of the remainder function in the collinear limit.

4 Interpretation of the collinear limit from the Regge Theory

The OPE expansion (46) for the remainder function has a form of the Fourier integral transform in the variable p. With the definition (39) for σ it is symmetric to the substitution $\sigma \to -\sigma$, which corresponds to the symmetry of the amplitude to the interchange of the cross ratios $u_1 \leftrightarrow u_3$. It is related to the symmetry of the Fourier transformed expression to the substitution $p \to -p$. Moreover, the function (46) can be analytically continued from the channel with $\sigma > 0$ to the channel with $\sigma < 0$ along the real axes, where it does not have any singularity. Note, that the channels with $\sigma > 0$ and $\sigma < 0$ are analogous to the s and u-channels for the elastic (nonplanar) amplitude.

However, in the attempt to continue (46) to the Mandelstam region with $u_1 \simeq |u_1|e^{-i2\pi}$, one faces some difficulties as it was discussed in the previous section. To overcome them we suggested to use another definition for σ (see (42))

$$\sigma \simeq \frac{1}{2}\ln\frac{u_1}{1-u_1}, \tag{76}$$

because in this case we could stay in the collinear limit with a fixed value of $\cos\phi$ in the course of the analytic continuation. Note, that the definitions (39) and (76) are equivalent in the Euclidean collinear region, where $u_3 \simeq 1-u_1$, but the use of (76) extends the region of applicability of the AGMSV remainder function R_{OPE} in (46). Note, that for the analytic continuation to the Mandelstam region with $u_3 \to 1$ one should use the symmetric definition $\sigma = 1/2\ln((1-u_3)/u_3)$.

It turns out, that we have an analogous situation with the variable

$$\tau \simeq \frac{1}{2} \ln \frac{4}{u_2}, \tag{77}$$

which tends to infinity in the collinear limit. Indeed, in the Regge kinematics $\sigma \simeq 1/2 \ln s_2 \to \infty$, where according to the definitions (3)

$$\tau \simeq \frac{1}{2} \ln \frac{4}{\tilde{u}_2} + \frac{1}{2} \ln \frac{1}{1-u_1} = \frac{1}{2} \ln \frac{4}{\tilde{u}_2} + \frac{1}{2} \ln \left(1 + e^{2\sigma}\right) \simeq \frac{1}{2} \ln \frac{4}{\tilde{u}_2} + \sigma \tag{78}$$

the variable τ depends on σ for fixed $\tilde{u}_2$, the expression for R_{OPE} in (46) would contain apart from the large terms of the order of $a \ln |w| \ln(1-u_1)$ also the comparatively large contributions $a \ln^2(1-u_1)$, which are not in an agreement with the double-logarithmic asymptotics (26) obtained from the BFKL resummation. In principle these contributions can be canceled by higher-loop corrections to the "coefficient function" $c^0(p)$. Indeed, at two loops the expression (45) for $h_1^{sub}(\sigma)$ contains the term σ^2 which cancels exactly a similar term in the leading contribution $\tau h_1(\sigma)$ in (43). In our opinion such miraculous cancelations can be avoided, if one would redefine τ appearing in the powers $\ell - 1$ in expressions of the type (29) in the following way

$$\tau \to \frac{1}{2} \ln \frac{u_3}{u_2} = -\ln |w|, \tag{79}$$

where $|w|$ is given in (10). In the collinear region this substitution can be justified in the LLA $a\tau \sim 1$, $a\sigma \ll 1$, where the corresponding OPE formula (29) was derived. Note, that the variables $|w|^{-2}$ and $1-u_1$ are analogous to the standard Bjorken variables $\mathbf{Q}^2$ and x in DIS. Thus, we suggest to write expression (46) in the collinear region matching the double-logarithmic limit as follows

$$R_{OPE} \simeq -a \cos(\phi_2 - \phi_3) \frac{|w|}{2} e^{-\sigma} \int_{-\infty}^{+\infty} dp \, e^{ip\sigma} c^0(p) (|w|^{a\gamma_1(p)} - 1), \tag{80}$$

where $\cos(\phi_2 - \phi_3)$ is defined in (10) and differs from $\cos\phi$ given by (38) only by the factor $\sqrt{u_1}$ and the overall sign.

Now we introduce the new variables

$$\sigma = \frac{1}{2} \ln \tilde{s}_2, \; \tilde{s}_2 = \frac{u_1}{1-u_1}, \; p = -i2\omega - i, \tag{81}$$

where $\tilde{s}_2$ is proportional to the invariant s_2 in Fig. 1. Note, that in the variable σ the remainder function at one or two loops contained an essential singularity at infinity. In the variable $\tilde{s}_2$ the essential singularity is absent.

Using (81) one can recast (80) to the Regge-like form

$$R_{OPE} \simeq \frac{a\pi}{4} \cos(\phi_2 - \phi_3) |w| \int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} \frac{d\omega}{2\pi i} \frac{(\tilde{s}_2)^\omega}{\omega(\omega+1)} \frac{1}{\sin \pi\omega} \left(|w|^{a\tilde{\gamma}_1(\omega)} - 1\right), \tag{82}$$

where $\omega = j - 1$ and

$$\tilde{\gamma}_1(\omega) = \gamma_1(-i2\omega - i) = \psi(2+\omega) + \psi(1-\omega) - 2\psi(1). \tag{83}$$

The function $e^\sigma R_{OPE}$ is symmetric to the substitution $\tilde{s}_2 \to 1/\tilde{s}_2$ and has singularities at the points (cf. Fig. 4)

$$\ln \tilde{s}_2 = \pm i\pi n \tag{84}$$

corresponding to the value $\tilde{s}_2 = (-1)^n$, where the integral in (82) is divergent at large ω. Due to the symmetry of $e^{\sigma} R_{OPE}$ to the substitution $\sigma \to -\sigma$ the points $\tilde{s}_2 = 0$ and $\tilde{s}_2 = -\infty$ are also singular and we can draw the cut in the $\tilde{s}_2$-plane from 0 to $-\infty$. The discontinuity of R_{OPE} on this cut has a singularity at the point $\tilde{s}_2 = -1$. Using formally its analytic continuation corresponding to the path **A** in Fig. 4 for large positive σ_0, we move along the large circle in a clockwise direction (see Fig. 5) in the $\tilde{s}_2$-plane and after crossing the cut at $\tilde{s}_2 < -1$ return to the initial point. The difference between the values of R_{OPE} after and before continuation in an accordance with the first equation of (54) is equal to the discontinuity on this cut, analytically continued from negative to the large positive $\tilde{s}_2$. We write the discontinuity before this continuation

$$\Delta R^{A}_{OPE}(-|\tilde{s}_2|) \simeq \frac{a\pi}{4} \cos(\phi_2 - \phi_3)|w|\theta(-1-\tilde{s}_2) \int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} \frac{d\omega}{2\pi i} \frac{(-2i)\ |\tilde{s}_2|^{\omega}}{\omega(\omega+1)} \left(|w|^{a\tilde{\gamma}_1(\omega)} - 1\right) \quad (85)$$

Note, that the discontinuity on the cut at $\tilde{s}_2 < -1$ is defined by a convergent integral, but its continuation $|\tilde{s}_2| \to e^{-i\pi}|\tilde{s}_2|$ to positive values of $\tilde{s}_2$ should be performed in a cautious way, because it demands the simultaneous rotation of the contour of integration by the angle π in anti-clockwise direction, which is a rather complicated procedure due to the infinite number of poles of the integrand. On the other hand, using the correct analytic continuation of R_{OPE}, corresponding to the path **B** in the σ-plane of Fig. 4, we initially cross the cut at $-1 < \tilde{s}_2 < 0$ moving from below and after that return to the initial point as illustrated in Fig. 5. In this case

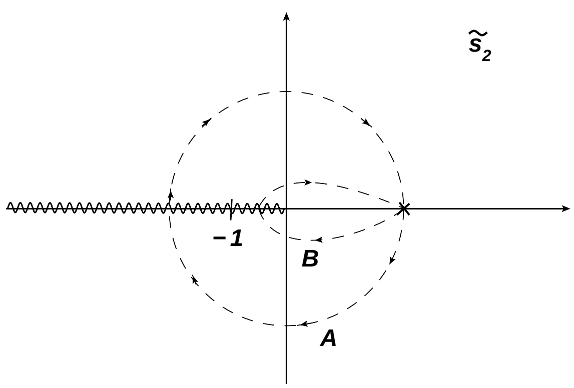

Figure 5: The paths **A** and **B** of the analytic continuation in the $\tilde{s}_2$-plane. The cross on the real axis denotes the initial point of the continuation. In the course of the analytic continuation along the path **B** we cross the branch cut on the real axis from 0 to -1 and return to the initial point. For the path **A** we cross the cut from -1 to $-\infty$.

the difference between the values of R_{OPE} after and before the continuation will be

$$\Delta R^{B}_{OPE}(-|\tilde{s}_2|) \simeq \frac{a\pi}{4} \cos(\phi_2 - \phi_3)|w|\theta(1+\tilde{s}_2)\theta(-\tilde{s}_2) \int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} \frac{d\omega}{2\pi i} \frac{(-2i)\ |\tilde{s}_2|^{\omega}}{\omega(\omega+1)} \left(|w|^{a\tilde{\gamma}_1(\omega)} - 1\right), (86)$$

analytically continued from negative to positive values of $\tilde{s}_2$. Again the discontinuity is given by a convergent integral at negative $\tilde{s}_2$, but its continuation $|\tilde{s}_2| \to e^{-i\pi}|\tilde{s}_2|$ requires a special consideration. The expressions (85) and (86) for discontinuities of the analytic continuations along the paths **A** and **B** are in an agreement with corresponding expressions for $h_k(\sigma)$ (see (54)). Note, that in both cases the multiplier $\sin \pi\omega$ in the denominator of the integrand is canceled. The factor $1/\sin \pi\omega$ can be considered as the usual signature factor in the Regge formulas allowing to obtain in the physical region of the t_2-channel the representation of amplitudes in terms of the Fourier sum of the partial wave contributions with positive integer values of ω. In

our case, however, we have a problem of returning to this Fourier-series representation because the t-channel partial waves have the additional pole at $\omega = 0$ and essential singularities at the points $\omega = 1, 2, 3, ...$ from the expansion of the exponent $|w|^{a\tilde{\gamma}_1(\omega)}$ in the series in powers of a. Note, that for amplitudes with the color-singlet quantum numbers in the t-channel such essential singularities are absent because the anomalous dimensions γ in this case do not have any poles at $\omega > 0$. The simplest example is the one-loop anomalous dimension in the $\mathcal{N} = 4$ SYM

$$\gamma^{singlet} = a(\psi(1) - \psi(\omega)). \tag{87}$$

Although the infrared divergencies in $\mathcal{N} = 4$ SYM could lead to the absence of the Fourier sum expansion in the physical region, the difference in the analytic properties of the t-channel partial waves in the ω-plane for the color singlet and adjoint representations looks strange. The question arises: whether or not one can construct the operator product expansion in the collinear limit in such a way, that the essential singularities of the partial waves in the corresponding semi-planes of the ω-plane would be absent. We see only one possibility of answering positively this question, namely, that the renormalization could not be multiplicative and there should be at least two operators having different anomalous dimension, which give comparable contributions in the collinear limit. To discuss this possibility, let us consider the dispersion representation for R_{OPE}

$$R_{OPE}(\tilde{s}_2) = \int_{-\infty}^{-1} \frac{d\tilde{s}_2'}{\pi(\tilde{s}_2' - \tilde{s}_2)} \frac{\Delta R^A\left(-|\tilde{s}_2'|\right)}{2i} + \int_{-1}^{0} \frac{d\tilde{s}_2'}{\pi(\tilde{s}_2' - \tilde{s}_2)} \frac{\Delta R^B\left(-|\tilde{s}_2'|\right)}{2i}, \tag{88}$$

where

$$\Delta R(-|\tilde{s}_2|) = R(e^{-i\pi}|\tilde{s}_2|) - R(e^{i\pi}|\tilde{s}_2|) \tag{89}$$

is given by expression

$$\frac{\Delta R^{A,B}}{2i} = \frac{a\pi}{4} \cos(\phi_2 - \phi_3)|w| \int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} \frac{d\omega}{2\pi i} |\tilde{s}_2|^{\omega} f_{\omega}^{A,B}(w). \tag{90}$$

In (90) we consider initially $f_{\omega}^{A,B}(w)$ only in two first orders of the perturbation theory

$$f_{\omega}^{A}(w) + f_{\omega}^{B}(w) = \frac{1}{\omega(\omega+1)}\left(1 + a\ln|w|\tilde{\gamma}_1(\omega)\right) \tag{91}$$

$$f_{\omega}^{A}(w) \simeq \frac{1}{\omega+1} + a\ln|w| \frac{\tilde{\gamma}_1^{-}(\omega) - 2\omega - 1}{\omega(\omega+1)}, \quad f_{\omega}^{B}(w) \simeq \frac{1}{\omega} + a\ln|w| \frac{\tilde{\gamma}_1^{+}(\omega) + 2\omega + 1}{\omega(\omega+1)}, \tag{92}$$

together with

$$\tilde{\gamma}_1^{+}(\omega) = \psi(1-\omega) - \psi(1), \quad \tilde{\gamma}_1^{-}(\omega) = \psi(\omega+2) - \psi(1), \tag{93}$$

where we took into account, that the partial wave $f_{\omega}^{A}(w)$ should not have singularities for $\omega > -1/2$ and $f_{\omega}^{B}(w)$ should not have singularities for $\omega < -1/2$ to provide vanishing ΔR^A in the region $|\tilde{s}_2| < 1$ as well as vanishing ΔR^B for $|\tilde{s}_2| > 1$. If we consider an analogy with the deep-inelastic $e-p$ scattering, the multiplicative renormalization takes place for the partial waves of the structure functions related directly to the imaginary part of the $\gamma^* p$ scattering amplitudes. In this case the momenta of the structure functions are proportional to the linear combination of matrix elements of the local operators (for integer ω). The local operators can mix each with others in the course of the renormalization and therefore in a general case they are

not renormalized in a multiplicative way. In the case of the remainder function in the collinear kinematics we also can expect that its discontinuities $\Delta R^{A,B}$ in the ω-plane are related to linear combinations of some operators. The comparatively simple situation will be if the number of the relevant operators is finite. In this case we can expect that OPE will be valid in the Mandelstam and other physical regions.

However, if we consider the discontinuities (85) and (86) for the collinear limit (80) of the remainder function, we obtain a very complicated result. Namely, $f_\omega^{A,B}(|w|)$ entering (90) are given by the expressions

$$f_\omega^A(|w|) = \int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} \frac{d\omega'}{2\pi i}\, \frac{1}{\omega'-\omega}\, \frac{1}{\omega'(\omega'+1)}\, e^{a\ln|w|\tilde{\gamma}_1(\omega')} \tag{94}$$

for $\Re\,(\omega) > -1/2$ and

$$f_\omega^B(|w|) = \int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} \frac{d\omega'}{2\pi i}\, \frac{1}{\omega'-\omega}\, \frac{1}{\omega'(\omega'+1)}\, e^{a\ln|w|\tilde{\gamma}_1(\omega')} \tag{95}$$

for $\Re\,(\omega) < -1/2$. Their sum is equal to the total partial wave

$$f_\omega^A(|w|) + f_\omega^B(|w|) = \frac{e^{a\ln|w|\tilde{\gamma}_1(\omega)}}{\omega(\omega+1)}. \tag{96}$$

However, $f_\omega^A(|w|)$ and $f_\omega^B(|w|)$ cannot be written as a finite sum of the exponential terms $e^{a\ln|w|\tilde{\gamma}_i(\omega)}$. Moreover, the simple renormalization properties will be absent in all Mandelstam regions obtained by the analytic continuation through the corresponding cuts in the $\tilde{s}_2$-plane. In our opinion this reflects some weakness of the simple exponentiation in the AGMSV remainder function. On the other hand, one can try to make an assumption, that the partial wave contains a sum of two exponents

$$f_\omega(|w|) = \frac{1}{\omega(\omega+1)} \left(e^{a\ln|w|\tilde{\gamma}_1^+(\omega)} + e^{a\ln|w|\tilde{\gamma}_1^-(\omega)} \right), \tag{97}$$

where the anomalous dimensions are given below

$$\tilde{\gamma}_1^+(\omega) = \psi(2+\omega) - \psi(1), \;\; \tilde{\gamma}_1^+(\omega) = \psi(1-\omega) - \psi(1), \;\; \tilde{\gamma}_1(\omega) = \tilde{\gamma}_1^+(\omega) + \tilde{\gamma}_1^-(\omega) \tag{98}$$

and contain the poles only in the left or in the right semiplanes of the ω-plane. At two loops the expressions in (96) and (97) coincide. At one loop they differ by a a factor of 2, but in the remainder function the one loop contribution should be subtracted. In principle we can subtract from (97) the term $1/\omega/(\omega+1)$ with the anomalous dimension equal to zero to reproduce the one-loop result.

For the ansatz (97) one can easily find the functions $f_\omega^A(|w|)$ and $f_\omega^B(|w|)$ for the discontinuities A and B

$$f_\omega^A(|w|) = \frac{1}{\omega(\omega+1)} \left(e^{a\ln|w|\tilde{\gamma}_1^+(\omega)} - (1+2\omega)e^{a\ln|w|} \right), \tag{99}$$

$$f_\omega^B(|w|) = \frac{1}{\omega(\omega+1)} \left(e^{a\ln|w|\tilde{\gamma}_1^-(\omega)} + (1+2\omega)e^{a\ln|w|} \right). \tag{100}$$

The function $f_\omega^A(|w|)$ is analytic for $\omega > -1/2$ and the function $f_\omega^B(|w|)$ is analytic for $\omega < -1/2$ in accordance to the fact, that $\Delta R^{A,B}$ are zero for $-1 < \tilde{s}_2 < 0$ and $\tilde{s}_2 < -1$, respectively.

Thus for the ansatz (97) we reproduce correctly the expression for the AGMSV remainder function R_{OPE} of Alday et al. [19] at two loops, but at higher loops the predictions are different. Note, that the expressions $f_\omega^A(|w|)$ and $f_\omega^B(|w|)$ contain the exponents for which the anomalous

dimension is the constant a. In the framework of the AdS/CFT correspondence the anomalous dimension is related to the energies of the string states in the Anti-de-Sitter space. Therefore, there should exist a string state in the adjoint representation, for which the energy does not depend on the angular momentum ω. Such a state should not have an inner structure and could be a gluon, which can be considered as an elementary particle, at least in our approximation.

Note, that both of the ansatz (96) and (97) are in agreement with the Regge asymptotics in the leading double-logarithmic approximation, as it was demonstrated in section 3.1. In that section we considered separate contributions from $\gamma_1^+(p)$ and $\gamma_1^-(p)$ in (59) to the leading logarithmic accuracy in the Mandelstam region. We found that the leading order BFKL result is fully reproduced if one takes into account only powers of $\gamma_1^-(p)$ in the AGMSV remainder function R_{OPE} in (46) up to five loops. $\gamma_1^+(p)$ contributes only at next-to-leading logarithmic level, which is not captured by the LLA BFKL analysis discussed in this study. This presents another argument in favor of the separate exponentiation of $\gamma_1^+(p)$ and $\gamma_1^-(p)$ in (97). To resolve the ambiguity in the different exponentiation prescriptions it is needed to calculate the Mandelstam cut contribution in the next-to-leading approximation, which will be hopefully obtained in the near future. In the conclusion we want to stress, that the ansatz (97) is the simplest one, which gives the finite superposition of the exponential terms $\propto e^{a \ln|w|\tilde{\gamma}_i(\omega)}$ for the discontinuities Δ^A and Δ^B. Therefore its verification in the next-to-leading BFKL calculation would be important.

5 Conclusions and discussions

In the present paper we studied the collinear and Regge limits of the $2 \to 4$ MHV amplitude. In particular we considered the analytic structure of the remainder function in the collinear kinematics proposed by Alday, Gaiotto, Maldacena, Sever and Vieira (AGMSV) and continued it analytically to the Mandelstam region. After the continuation, the AGMSV expression in the Regge limit reproduces the BFKL results in the double-logarithmic approximation up to five-loop level. However, we also note that all of the contributions reproducing the known BFKL expressions can be obtained from only one piece of the anomalous dimension $\gamma_1(p)$ (see (30)) present in the AGMSV formula. This piece $\gamma_1^-(p)$, defined by (59), has singularities only in the lower semiplane of the complex p-plane. This translates into the right singularities in the complex angular momentum plane as discussed in section 4.

In the Regge theory one can expect a clear separation between the right and the left singularities in the complex angular momentum plane, suggesting a non-multiplicative renormalization of the remainder function in the Euclidean region of the collinear kinematics. In other words there could be at least two operators having different anomalous dimensions. This gives the same result as a simple one-operator renormalization at two loops. The difference between the simple renormalization of the AGMSV expression and the two-operator renormalization suggested in section 4 appears already at 3 loops and can be verified only by taking into account next-to-leading corrections to the BFKL eigenvalue in the adjoint color representation. These can be extracted from the NLO BFKL Kernel calculated by Fadin and Fiore [25, 26] and will hopefully be found in the near future.

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Appendix

A Double Leading Logarithmic Approximation (DLLA)

We consider the Double Leading Logarithmic Approximation (DLLA) for the remainder function calculated in the BFKL approach. We start with the remainder function in the Leading Logarithmic Approximation (LLA) given by (7) and (8), where we omit all terms subleading in the logarithm of the energy $\ln s_2 \simeq -\ln(1-u_1)$. This expression was calculated in the multi-Regge kinematics given by (2). Imposing an additional kinematic constraint, that corresponds to the collinear limit (21) we expand (8) in powers of $|w|$ in accordance with (22). The leading contribution comes from the conformal spin $n=\pm 1$, and the only the first term in $E_{\nu,n}$ of (13) has the relevant poles. We can approximate

$$E_{\nu,n} \simeq E_{\nu,1} \simeq -\frac{1}{2}\frac{1}{\nu^2+\frac{1}{4}} \tag{A.1}$$

in the double logarithmic approximation and write

$$R^{DLLA}_{BFKL} \simeq 1 - i\frac{a}{2}\left(\frac{w}{w^*}+\frac{w^*}{w}\right)\sum_{k=1}^{\infty}\frac{a^k \ln^k(1-u_1)}{k!}\int_{-\infty}^{\infty}\frac{d\nu\; |w|^{2i\nu}}{\nu^2+\frac{1}{4}}E^k_{\nu,1} \tag{A.2}$$

$$\simeq 1 - ia\cos(\phi_2-\phi_3)\sum_{k=1}^{\infty}\frac{(-1)^k a^k 2^{-k}\ln^k(1-u_1)}{k!}\int_{-\infty}^{\infty}\frac{d\nu\; |w|^{2i\nu}}{\left(\nu^2+\frac{1}{4}\right)^{k+1}} \tag{A.3}$$

$$\simeq 1 - \; i2\pi a\cos(\phi_2-\phi_3)\;|w|\sum_{k=1}^{\infty}\frac{a^k \ln^k|w|\ln^k(1-u_1)}{(k!)^2} \tag{A.4}$$

$$= 1 + i2\pi a\cos(\phi_2-\phi_3)\;|w|\left(1 - I_0\left(2\sqrt{a\ln|w|\ln(1-u_1)}\right)\right), \tag{A.5}$$

where $I_0(z)$ is the modified Bessel function.

The contribution to the real part of the NLLA remainder function comes from several terms in the dispersion-like relation in (16). Expanding (16) to the second and the third order in powers of a we obtain

$$a^2 R^{(2)} - \frac{\pi^2\delta^2}{2} = -\frac{\pi^2\omega_{ab}^2}{2} + i\frac{a^2}{2}\frac{\partial^2}{\partial a^2}\left(\int_{-i\infty}^{i\infty}\frac{d\omega}{2\pi i}f(\omega)e^{-i\pi\omega}|1-u_1|^{-\omega}\right) \tag{A.6}$$

and

$$a^3 R^{(3)} + i\pi\delta a^2 R^{(2)} - i\frac{\pi^3\delta^3}{6} = i\frac{a^3}{6}\frac{\partial^3}{\partial a^3}\left(\int_{-i\infty}^{i\infty}\frac{d\omega}{2\pi i}f(\omega)e^{-i\pi\omega}|1-u_1|^{-\omega}\right). \tag{A.7}$$

We are interested only in the leading logarithmic (LLA) and the real part of the next-to-leading (NLLA) in the logarithm $\ln(1-u_1)$ contributions. Thus we can omit all subleading terms in (A.6) and (A.7) as follows

$$a^2 R^{(2)}_{BFKL} - \frac{\pi^2\delta^2}{2} = -\frac{\pi^2\omega_{ab}^2}{2} + i\frac{a^2}{2}\frac{\partial^2}{\partial a^2}\left(\int_{-i\infty}^{i\infty}\frac{d\omega}{2\pi i}f^{LLA}(\omega)e^{-i\pi\omega}|1-u_1|^{-\omega}\right) \tag{A.8}$$

and

$$a^3 R^{(3)}_{BFKL} + i\pi\delta a^2 R^{(2)}_{BFKL} = i\frac{a^3}{6}\frac{\partial^3}{\partial a^3}\left(\int_{-i\infty}^{i\infty}\frac{d\omega}{2\pi i}f^{LLA}(\omega)e^{-i\pi\omega}|1-u_1|^{-\omega}\right). \tag{A.9}$$

The integral in RHS of (A.8) and (A.9) is related to Δ defined in (8) by

$$\Delta^{LLA}_{2\to 4} = \int_{-i\infty}^{i\infty}\frac{d\omega}{2\pi i}f^{LLA}(\omega)|1-u_1|^{-\omega} \tag{A.10}$$

for $f^{LLA}(\omega)$ given by (18), and was calculated to the second [7] and the third order [16] in a. The phase of its integrand $e^{-i\pi\omega}$ can be accounted for by making a substitution $\ln(1-u_1) \to \ln(1-u_1) + i\pi$ in the final result. At two loops we have a full cancellation between real NLLA contributions coming from the integral, δ and ω_{ab}. This is not the case at three loops, where the real part of the NLLA remainder function was calculated in ref. [16] and reads

$$R_{BFKL}^{(3)\,LLA} = i\Delta_{2\to4}^{(3)}/a^3 = i\pi\frac{1}{4}\ln^2(1-u_1)\left(\ln|w|^2\ln^2|1+w|^2 - \frac{2}{3}\ln^3|1+w|^2\right. \tag{A.11}$$
$$\left.-\frac{1}{4}\ln^2|w|^2\ln|1+w|^2 + \frac{1}{2}\ln|w|^2\left(\mathrm{Li}_2(-w)+\mathrm{Li}_2(-w^*)\right) - \mathrm{Li}_3(-w) - \mathrm{Li}_3(-w^*)\right).$$

Using dispersion-like relation (16), it is possible to write a general relation between the LLA and the real part of NLLA remainder function at an arbitrary number of loops

$$\Re\left(R_{BFKL}^{NLLA}\right) = 1 + i\pi\frac{\partial R_{BFKL}^{(LLA)}}{\partial\ln(1-u_1)} - i\pi\delta R_{BFKL}^{(LLA)} + \frac{\pi^2}{2}\left(\delta^2-\omega_{ab}^2\right), \tag{A.12}$$

where δ and ω_{ab} are given by (17), and the integral representation for $R_{BFKL}^{(LLA)}$ is known and can be analytically calculated to any order.

In the double logarithmic approximation (DLLA), where both $1-u_1$ and $|w|$ are small, the relation (A.12) can be explicitly calculated as follows. First, we find

$$\frac{\pi^2}{2}\left(\delta^2-\omega_{ab}^2\right) = \frac{a^2\pi^2}{2}\ln|1+w|^2\ln\left|1+\frac{1}{w}\right|^2 \simeq -2\pi^2a^2\cos(\phi_2-\phi_3)|w|\ln|w|, \tag{A.13}$$

next, using (A.2) we readily obtain

$$-i\pi\delta R_{BFKL}^{(LLA)} \simeq 4\pi^2 a\cos(\phi_2-\phi_3)\;|w|\;\ln|w|\left(1 - I_0\left(2\sqrt{a\ln|w|\ln(1-u_1)}\right)\right). \tag{A.14}$$

Finally, the first term in RHS of (A.12) can be calculated replacing $\ln^k(1-u_1)$ in (A.2) by $i\pi k\ln^{k-1}(1-u_1))$, because of the phase of the integrand $e^{-i\pi\omega}$ in the dispersion relation (16) generates terms with $\ln(1-u_1) \to \ln(1-u_1)+i\pi$ and we are interested only in the next-to-leading contributions

$$i\pi\frac{\partial R_{BFKL}^{(LLA)}}{\partial\ln(1-u_1)} = a\;2\pi^2\cos(\phi_2-\phi_3)\;|w|\sum_{k=1}^{\infty}\frac{a^k\ln^k|w|k\ln^{k-1}(1-u_1)}{(k!)^2} \tag{A.15}$$
$$= 2\pi^2a^{3/2}\cos(\phi_2-\phi_3)\;|w|\;\ln|w|\frac{I_1\left(2\sqrt{a\ln|w|\ln(1-u_1)}\right)}{\ln(1-u_1)}.$$

Plugging (A.13), (A.14) and (A.15) in (A.12) we get in the double logarithmic approximation the real part of the contribution subleading in $\ln(1-u_1)$

$$\Re\left(R_{BFKL}^{DNLLA}\right) \simeq 1 + 2\pi^2a^{3/2}\cos(\phi_2-\phi_3)\;|w|\;\ln|w|\frac{I_1\left(2\sqrt{a\ln|w|\ln(1-u_1)}\right)}{\ln(1-u_1)} \tag{A.16}$$
$$+4\pi^2a\cos(\phi_2-\phi_3)\;|w|\;\ln|w|\left(1-I_0\left(2\sqrt{a\ln|w|\ln(1-u_1)}\right)\right) - 2\pi^2a^2\cos(\phi_2-\phi_3)|w|\ln|w|.$$

B The AGMSV remainder function at three loops

In this section we present details of the calculation of the three loop contribution to the remainder function in the collinear limit given by (46). At three loops it reads

$$R_{OPE}^{(3)} \simeq \cos\phi\;e^{-\tau}\frac{\tau^2}{2}\int c^0(p)\gamma_1^2(p)e^{ip\sigma}dp, \tag{B.1}$$

where

$$c^0(p) = \frac{2}{1+p^2}\frac{1}{\cos\frac{p\pi}{2}} \tag{B.2}$$

and

$$\gamma_1(p) = \psi\left(\frac{3}{2}+i\frac{p}{2}\right)+\psi\left(\frac{3}{2}-i\frac{p}{2}\right)-2\psi\left(1\right). \tag{B.3}$$

In order to find $R^{(3)}_{OPE}$ we need to calculate

$$h_2(\sigma) = \int_{-\infty}^{\infty} c^0(p)\gamma_1^2(p)e^{ip\sigma}dp \tag{B.4}$$

defined in (48). At two loops the remainder function $R^{(2)}_{OPE}$ in the collinear limit was found in ref. [19] and the relevant integral reads

$$h_1(\sigma) = \int_{-\infty}^{\infty} c^0(p)\gamma_1(p)e^{ip\sigma}dp, \tag{B.5}$$

where

$$h_1(\sigma) = -2\cosh\sigma\left(2\ln\left(1+e^{2\sigma}\right)\ln\left(1+e^{-2\sigma}\right)-4\ln\left(2\cosh\sigma\right)\right)-8\sigma\sinh\sigma. \tag{B.6}$$

From (B.6) we see that the most complicated term is given by $\cosh\sigma\ln^2\left(2\cosh\sigma\right)$. At three loops it is natural to expect $\cosh\sigma\ln^3\left(2\cosh\sigma\right)$, but this function diverges at $\sigma\to\pm\infty$ and need to be cured by $\sigma^3\sinh\sigma$ term. Other way to cure the divergency is to introduce the regularization parameter $p\to p-i\epsilon$ in the integral

$$\int_{-\infty}^{\infty}\cosh\sigma\ln^3\left(2\cosh\sigma\right)e^{-i(p-i\epsilon)\sigma}d\sigma = \frac{\partial^3}{\partial a^3}\frac{1}{2}\int_0^{\infty}\left(z+\frac{1}{z}\right)^a z^{-i(p-i\epsilon)-1}dz|_{a=0}, \tag{B.7}$$

where $z=e^\sigma$. The last integral in (B.7) gives the Euler Beta function and its higher derivatives that give the polygamma functions. In an analogous way we calculate the term $\sigma^3\sinh\sigma$ and obtain the final expression with no ϵ dependence

$$\begin{aligned}&8\left(\cosh\sigma\ln^3\left(2\cosh\sigma\right)-\sigma^3\sinh\sigma\right)\\&=\int_{-\infty}^{\infty}c^0(p)\left(-6\gamma_1(p)+\frac{3}{2}\gamma_1^2(p)+\frac{3}{2}\tilde{\gamma}_1(p)+\frac{48}{(1+p^2)^2}-\frac{24}{1+p^2}-\frac{\pi^2}{2}\right)e^{ip\sigma}dp,\end{aligned} \tag{B.8}$$

where

$$\tilde{\gamma}_1(p) = \psi'\left(\frac{3}{2}+i\frac{p}{2}\right)+\psi'\left(\frac{3}{2}-i\frac{p}{2}\right)-2\psi'\left(1\right). \tag{B.9}$$

and the functions $\gamma_1(p)$ and $c^0(p)$ are given by (B.3) and (B.2) respectively. The first term on RHS of (B.8) is proportional to $h_1(p)$ and the last term is known from ref. [19]

$$h_0(\sigma) = \int_{-\infty}^{\infty}c^0(p)e^{ip\sigma}dp = 4\cosh\sigma\ln(2\cosh\sigma)-4\sigma\sinh\sigma. \tag{B.10}$$

The rest of the terms in (B.9) are calculated using the Cauchy theorem. For simplicity we consider only the case of positive σ closing the integration contour in the upper semiplane. The

terms in (B.9) have a higher order pole at $p=i$ and all other poles are simple poles. Thus we can readily write

$$\begin{aligned} I_1 &= \int_{-\infty}^{\infty} c^0(p)\frac{1}{(1+p^2)^2}e^{ip\sigma}dp \qquad \text{(B.11)}\\ &= \frac{e^{-\sigma}}{2}\left(\frac{5}{2}+\frac{\pi^2}{8}+3\sigma+\frac{\pi^2\sigma}{12}+\frac{3\sigma^2}{2}+\frac{\sigma^3}{3}\right)-\sum_{n=1}^{\infty}\frac{(-1)^n e^{-\sigma(1+2n)}}{8n^3(1+n)^3}\\ &= \frac{e^{-\sigma}}{2}\left(\frac{\pi^2}{8}+\frac{\pi^2\sigma}{12}+\frac{3\sigma^2}{2}+\frac{\sigma^3}{3}\right)+\frac{3}{8}h_0-\frac{3}{4}\sinh\sigma\ \mathrm{Li}_2(-e^{-2\sigma})-\frac{1}{4}\cosh\sigma\ \mathrm{Li}_3(-e^{-2\sigma}), \end{aligned}$$

where the first term on RHS comes from the pole at $p=i$.

The transform of $c^0(p)/(1+p^2)$ can be obtained directly from (B.11) differentiating it twice with respect to σ

$$I_2 = \int_{-\infty}^{\infty} c^0(p)\frac{1}{1+p^2}e^{ip\sigma}dp = -\frac{d^2I_1}{d\sigma^2}+I_1 = \frac{\pi^2e^{-\sigma}}{12}+\sigma^2e^{-\sigma}+\frac{h_0}{2}-\sinh\sigma\ \mathrm{Li}_2(-e^{-2\sigma}) \qquad \text{(B.12)}$$

Finally we calculate the last missing contribution in (B.8)

$$\begin{aligned} &\int_{-\infty}^{\infty} c^0(p)\tilde{\gamma}_1(p)e^{ip\sigma}dp = -6e^{-\sigma}-4\sigma e^{-\sigma}-\sum_{n=1}^{\infty}\frac{(-1)^n 2e^{-\sigma(1+2n)}}{n(1+n)}\left(\frac{1+2\sigma}{n(n+1)}\zeta_2+\frac{4\sigma}{n+1}+2\sigma^3\right)\\ &= -\frac{\pi^2e^{-\sigma}}{3}-4\sigma^2e^{-\sigma}+2\cosh\sigma\left(-\frac{\pi^2}{3}-4\sigma^3-4\ln(2\cosh\sigma)+\frac{\pi^2}{3}\ln(2\cosh\sigma)\right. \qquad \text{(B.13)}\\ &\left.+4\sigma^2\ln(2\cosh\sigma)-4\mathrm{Li}_2(-e^{-2\sigma})\right)+2\pi\sinh\sigma\left(2\sigma+\mathrm{Li}_2(-e^{-2\sigma})\right) \end{aligned}$$

This allows us to find the integral in the three loop expression (B.1)

$$\begin{aligned} h_2(\sigma) &= \int_{-\infty}^{\infty} c^0(p)\gamma_1^2(p)e^{ip\sigma}dp = -\frac{\pi^2}{3}e^{-\sigma}-4e^{-\sigma}\sigma^2-\frac{2}{3}\pi^2\sigma\cosh\sigma \qquad \text{(B.14)}\\ &+16\sigma^2\cosh\sigma+\frac{8}{3}\sigma^3\cosh\sigma+24\cosh\sigma\ln(2\cosh\sigma)+\frac{2}{3}\pi^2\cosh\sigma\ln(2\cosh\sigma)\\ &-8\sigma^2\cosh\sigma\ln(2\cosh\sigma)-16\cosh\sigma\ln^2(2\cosh\sigma)+\frac{16}{3}\cosh\sigma\ln^3(2\cosh\sigma)\\ &+8\sigma\cosh\sigma\ \mathrm{Li}_2(-e^{-2\sigma})+8\cosh\sigma\ \mathrm{Li}_3(-e^{-2\sigma})-24\sigma\sinh\sigma+4\sinh\sigma\ \mathrm{Li}_2(-e^{-2\sigma}) \end{aligned}$$

The expression in (B.14) vanishes at $\sigma\to\infty$ and is symmetric under $\sigma\to-\sigma$.

C Analytic continuation

In this section we perform the analytic continuation of the AGMSV remainder function R_{OPE} in (46). The analytic continuation (u_2, u_3 are fixed and $u_1=|u_1|e^{-i2\pi}$), which was used [15, 16] to extract (14) and (15) from the GSVV remainder function, is not applicable here. This is because the collinear function R_{OPE} was obtained under assumption of the finiteness of the cosine factor

$$\cos\phi = \frac{u_1+u_2+u_3-1}{2\sqrt{u_1u_2u_3}}, \qquad \text{(C.1)}$$

which diverges when we cross a point $u_1=|u_1|e^{-i\pi}$ in the multi-Regge kinematics given by (2). At this point the numerator becomes of the order of unity, while the denominator is small.

Using the parametrization of the dual conformal cross ratios introduced in ref. [19] and given by (28) we find

$$\frac{u_1}{u_3} = e^{2\sigma}, \tag{C.2}$$

so that the path of the continuation in the complex σ-plane is just a shift of σ, namely

$$\sigma \Rightarrow \sigma - i\pi. \tag{C.3}$$

We call this path of the analytic continuation- *the path* **A**.

It was argued in section 3, that one can smoothly deform the path of the continuation to make it compatible with the collinear limit by taking into account the relation

$$u_3 \to 1 - u_1 \tag{C.4}$$

for $u_2 \to 0$. This makes the numerator of $\cos\phi$ to be of the order of its denominator at $u_1 = |u_1|e^{-i\pi}$ and thus to be compatible with the basic assumptions of the collinear expansion in τ. We can define

$$e^{2\sigma} \simeq \frac{u_1}{1-u_1}, \quad \sigma \simeq \frac{1}{2}\ln\frac{u_1}{1-u_1} \tag{C.5}$$

for $u_1 = |u_1|e^{-i\pi}$ as the analytic continuation along *the path* **B**. In the continuation with the path **B** the cross ratio u_3 is not fixed anymore and possesses a non-trivial phase as u_1 rotates around the origin. The paths **A** and **B** in the complex σ-plane are illustrated in Fig. 4. Here list the analytic continuation along the path **B** of the functions relevant at two and three loops. By the words "Regge limit" over the arrows we mean the multi-Regge and collinear kinematics (21) for which $\sigma \simeq -1/2\ln(1-u_1) \to +\infty$.

$$\sigma \simeq \frac{1}{2}\ln u_1 - \frac{1}{2}\ln(1-u_1) \Rightarrow -i\pi + \sigma \xrightarrow{\text{Regge limit}} -i\pi - \frac{1}{2}\ln(1-u_1) \tag{C.6}$$

$$e^{\sigma} \simeq \sqrt{\frac{u_1}{1-u_1}} \Rightarrow -e^{\sigma} \xrightarrow{\text{Regge limit}} \frac{-1}{\sqrt{1-u_1}}, \quad e^{-\sigma} \simeq \sqrt{\frac{1-u_1}{u_1}} \Rightarrow -e^{-\sigma} \xrightarrow{\text{Regge limit}} 0, \tag{C.7}$$

$$\cosh\sigma \Rightarrow -\cosh\sigma \xrightarrow{\text{Regge limit}} \frac{-1}{2\sqrt{1-u_1}}, \ \sinh\sigma \Rightarrow -\sinh\sigma \xrightarrow{\text{Regge limit}} \frac{-1}{2\sqrt{1-u_1}} \tag{C.8}$$

$$\ln(2\cosh\sigma) \simeq \ln\left(\frac{1}{\sqrt{u_1(1-u_1)}}\right) \Rightarrow i\pi + \ln(2\cosh\sigma) \xrightarrow{\text{Regge limit}} i\pi - \frac{1}{2}\ln(1-u_1) \tag{C.9}$$

$$\begin{aligned} \mathrm{Li}_2(-e^{-2\sigma}) &= \mathrm{Li}_2\left(\frac{u_1-1}{u_1}\right) = -\int_0^{\frac{u_1-1}{u_1}} \frac{dt}{t}\ln(1-t) \Rightarrow \mathrm{Li}_2(-e^{-2\sigma}) - i2\pi\int_1^{\frac{u_1-1}{u_1}} \frac{dt}{t} \\ &= \mathrm{Li}_2(-e^{-2\sigma}) - i2\pi(-2\sigma + i\pi) \xrightarrow{\text{Regge limit}} -i2\pi(\ln(1-u_1) + i\pi) \end{aligned} \tag{C.10}$$

$$\mathrm{Li}_3(-e^{-2\sigma}) \Rightarrow \mathrm{Li}_3(-e^{-2\sigma}) - i2\pi \int_1^{\frac{u_1-1}{u_1}} \frac{dt}{t} \int_1^t \frac{dt'}{t'} = \mathrm{Li}_3(-e^{-2\sigma}) - i2\pi \frac{(-2\sigma+i\pi)^2}{2}$$

$$\overset{\text{Regge limit}}{\overbrace{\longrightarrow}} \quad -i2\pi \frac{(\ln(1-u_1)+i\pi)^2}{2} \tag{C.11}$$

Having the list of all necessary functions continued along the path **B**, we can readily write the analytic continuation of $h_k(\sigma)$ defined by (48). Note that the general structure of $h_k(\sigma)$ after the analytic continuation is

$$h_k(\sigma) \Rightarrow -h_k(\sigma) + \Delta_k(\sigma), \tag{C.12}$$

where $h_k(\sigma)$ changes the sign and receives an additive function. The change of the sign is related to the fact that $\cos\phi$ also changes the sign

$$\cos\phi = \frac{u_1+u_2+u_3-1}{2\sqrt{u_1u_2u_3}} \Rightarrow -\cos\phi \quad \overset{\text{Regge limit}}{\overbrace{\longrightarrow}} \quad -\frac{u_1+u_2+u_3-1}{2\sqrt{u_2u_3}} = \cos(\phi_2-\phi_3) \tag{C.13}$$

so that the product of $\cos\phi$ and $h_k(\sigma)$ that appears in the AGMSV remainder function in (46) has the same sign and gets an additive function after the analytic continuation. We start with $h_0(\sigma)$ in (35), which corresponds to one loop, i.e. the BDS amplitude

$$h_0(\sigma) \Rightarrow -h_0(\sigma) - i4\pi e^{\sigma} \quad \overset{\text{Regge limit}}{\overbrace{\longrightarrow}} \quad \frac{-i4\pi}{\sqrt{1-u_1}} \tag{C.14}$$

and read out of it

$$\Delta_0(\sigma) = -i4\pi e^{\sigma}. \tag{C.15}$$

Next the we consider $h_1(\sigma)$ for the AGMSV remainder function at two loops

$$h_1(\sigma) \Rightarrow -h_1(\sigma) - i8\pi e^{\sigma} + i8\pi\cosh\sigma\ln(1+e^{2\sigma}) \quad \overset{\text{Regge limit}}{\overbrace{\longrightarrow}} \quad -\frac{i4\pi}{\sqrt{1-u_1}}\ln(1-u_1) - \frac{i8\pi}{\sqrt{1-u_1}}, \tag{C.16}$$

which gives

$$\Delta_1(\sigma) = -i8\pi e^{\sigma} + i8\pi\cosh\sigma\ln(1+e^{2\sigma}). \tag{C.17}$$

At three loops we have

$$\begin{aligned} &h_2(\sigma) \Rightarrow -h_2(\sigma) - i24\pi e^{\sigma} + 4ih_1(\sigma)\pi - 4\pi^2 e^{\sigma} + 2\pi^2 h_0(\sigma) + i24\pi e^{\sigma}\sigma \qquad \text{(C.18)}\\ &-4i\pi h_0(\sigma)\sigma + 8\pi^2 e^{\sigma}\sigma + \frac{4}{3}i\pi^3\cosh\sigma + i8\pi\cosh\sigma\mathrm{Li}_2(-e^{-2\sigma}) - i16\pi\sigma^2\sinh\sigma \quad \overset{\text{Regge limit}}{\overbrace{\longrightarrow}}\\ &-\frac{i24\pi}{\sqrt{1-\mathrm{u1}}} - \frac{4\pi^2}{\sqrt{1-\mathrm{u1}}} + \frac{i2\pi^3}{3\sqrt{1-\mathrm{u1}}} - \frac{i12\pi\ln(1-u_1)}{\sqrt{1-\mathrm{u1}}} - \frac{4\pi^2\ln(1-u_1)}{\sqrt{1-\mathrm{u1}}} - \frac{i2\pi\ln^2(1-u_1)}{\sqrt{1-\mathrm{u1}}}, \end{aligned}$$

from which we extract

$$\begin{aligned} \Delta_2(\sigma) &= 8i\pi\cosh\sigma\mathrm{Li}_2\left(-e^{-2\sigma}\right) + 16i\pi\sigma^2\cosh\sigma - 8i\pi e^{\sigma}\sigma - 4\pi^2 e^{\sigma} - 24i\pi e^{\sigma} \qquad \text{(C.19)}\\ &+\frac{4}{3}i\pi^3\cosh\sigma - 16i\pi\cosh\sigma\ln^2\left(1+e^{2\sigma}\right) + 16i\pi\sigma\cosh\sigma\ln\left(1+e^{2\sigma}\right)\\ &+8\pi^2\cosh\sigma\ln\left(1+e^{2\sigma}\right) + 32i\pi\cosh\sigma\ln\left(1+e^{2\sigma}\right). \end{aligned}$$

The AGMSV remainder function in (46) at two loops reads

$$R^{(2)}_{OPE} \simeq -\cos\phi\; e^{-\tau}\tau \int_{-\infty}^{\infty} c^0(p)\gamma_1(p) e^{ip\sigma} dp = -\cos\phi\; e^{-\tau}\;\tau\; h_1(\sigma) \tag{C.20}$$

and after the analytic continuation along path **B** we obtain (note that $\cos\phi \Rightarrow -\cos\phi$ after the analytic continuation)

$$R^{(2)}_{OPE} \Rightarrow \cos\phi\; e^{-\tau}\;\tau\;(-h_1(\sigma) - i8\pi e^{\sigma} + i8\pi\cosh\sigma\ln(1+e^{2\sigma})) \tag{C.21}$$

$$\overset{\text{Regge limit}}{\underset{\sigma\to+\infty}{\overbrace{\longrightarrow}}} \;-\cos(\phi_2-\phi_3)\frac{\sqrt{u_2}}{2}(\ln 2 - \ln\sqrt{u_2})\left(-\frac{i4\pi}{\sqrt{1-u_1}}\ln(1-u_1) - \frac{i8\pi}{\sqrt{1-u_1}}\right)$$

$$= i2\pi\cos(\phi_2-\phi_3)\;|w|\;\left(\ln 2 - \ln|w| - \frac{1}{2}\ln(1-u_1)\right)(\ln(1-u_1)+2)\,, \tag{C.22}$$

where we used the definition $|w|^2 = u_2/u_3$ and the fact that in the collinear limit we have

$$u_3 \to 1-u_1 \tag{C.23}$$

as well as

$$e^{-\tau} \simeq \frac{\sqrt{u_2}}{2} = \frac{|w|\sqrt{1-u_1}}{2},\;\; \tau \simeq \ln 2 - \ln\sqrt{u_2} = \ln 2 - \ln|w| - \frac{1}{2}\ln(1-u_1). \tag{C.24}$$

Already at this point we see that the terms leading in $\ln|w|$ reproduce the BFKL result. However, in order to find the full agreement we have to include the terms subleading in τ because their smallness is of the same order as those enhanced by $\ln(1-u_1)$. These we extract from the GSVV expression

$$\begin{aligned} h_1^{sub}(\sigma) &= -\frac{2}{3}\pi^2\sigma\cosh\sigma - 4\sigma^2\cosh\sigma - \frac{8}{3}\sigma^3\cosh\sigma + 4\sigma^2\cosh\sigma\ln 2 \\ &-8\cosh\sigma\ln(2\cosh\sigma) + \frac{2}{3}\pi^2\cosh\sigma\ln(2\cosh\sigma) + 4\sigma^2\cosh\sigma\ln(2\cosh\sigma) \\ &+8\cosh\sigma\ln(2\cosh\sigma)\ln 2 + 4\cosh\sigma\ln^2(2\cosh\sigma) - 4\cosh\sigma\ln^2(2\cosh\sigma)\ln 2 \\ &-\frac{4}{3}\cosh\sigma\ln^3(2\cosh\sigma) + 4\cosh\sigma\,\mathrm{Li}_3\left(-e^{-2\sigma}\right) - 8\sigma\sinh\sigma - 8\sigma\sinh\sigma\ln 2, \end{aligned} \tag{C.25}$$

so that for $\tau \to +\infty$ one can write

$$R^{(2)}_{GSVV} \simeq \cos\phi\; e^{-\tau}\left(-\tau h_1(\sigma) + h_1^{sub}(\sigma)\right) + \mathcal{O}\left(e^{-2\tau}\right). \tag{C.26}$$

After the analytic continuation this gives

$$h_1^{sub}(\sigma) \Rightarrow -h_1^{sub}(\sigma) + \Delta_1^{sub}(\sigma), \tag{C.27}$$

where

$$\begin{aligned} \Delta_1^{sub}(\sigma) &= 8i\pi e^{\sigma} - 8i\pi e^{\sigma}\ln 2 + 4i\pi\cosh\sigma\ln^2\left(1+e^{2\sigma}\right) \\ &-8i\pi\cosh\sigma\ln\left(1+e^{2\sigma}\right) + 8i\pi\ln 2\cosh\sigma\ln\left(1+e^{2\sigma}\right) \end{aligned} \tag{C.28}$$

and the relevant terms then become

$$\cos\phi\; e^{-\tau} h_1^{sub}(\sigma) \Rightarrow -\cos\phi\; e^{-\tau}\left(-h_1^{sub}(\sigma) + \Delta_1^{sub}(\sigma)\right) \overset{\text{Regge limit}}{\overbrace{\longrightarrow}}$$

$$\cos(\phi_2-\phi_3)\frac{\sqrt{u_2}}{2}\left(\frac{8i\pi}{\sqrt{1-u_1}} - \frac{8i\pi\ln 2}{\sqrt{1-u_1}} + \frac{4i\pi\ln(1-u_1)}{\sqrt{1-u_1}} - \frac{4i\pi\ln 2\ln(1-u_1)}{\sqrt{1-u_1}} + \frac{2i\pi\ln(1-u_1)^2}{\sqrt{1-u_1}}\right)$$

$$= i\pi\cos(\phi_2-\phi_3)\;|w|\;\left(4 - 4\ln 2 + 2\ln(1-u_1) - 2\ln 2\ln(1-u_1) + \ln^2(1-u_1)\right).$$

Adding this to the AGMSV remainder function we obtain

$$R^{(2)}_{OPE} + \cos\phi \, e^{-\tau} h_1^{sub}(\sigma) \Longrightarrow -i2\pi \cos(\phi_2 - \phi_3) \, |w| \, (\ln(1-u_1) \ln|w| + 2\ln|w| - 2), \quad \text{(C.29)}$$

which fully reproduces the BFKL result and the subleading corrections extracted from the GSVV expression given by (23) in the collinear limit.

Applying a similar analysis to the AGMSV remainder function at three loops we also reproduce the BFKL result in the collinear and multi-Regge kinematics given by (21) as follows. After the analytic continuation of $h_2(\sigma)$ along the path **B** we have

$$h_2(\sigma) \Rightarrow -h_2(\sigma) + \Delta_2(\sigma), \quad \text{(C.30)}$$

where $\Delta_2(\sigma)$ is given by (C.19). Plugging this in the expression for the remainder function in the collinear limit (46) we obtain

$$\begin{aligned}
& R^{(3)}_{OPE} \simeq \cos\phi \, e^{-\tau} \, \frac{\tau^2}{2} \, h_2(\sigma) \Rightarrow -\cos\phi e^{-\tau} \frac{\tau^2}{2} \left(-h_2(\sigma) + \Delta_2(\sigma)\right) \overset{\text{Regge limit}}{\longrightarrow} \quad \text{(C.31)} \\
& \cos(\phi_2 - \phi_3) \frac{\sqrt{u_2}}{2} \frac{\left(-\frac{1}{2}\ln u_2 + \ln 2\right)^2}{2} \left(-\frac{i24\pi}{\sqrt{1-\mathrm{u1}}} - \frac{4\pi^2}{\sqrt{1-\mathrm{u1}}} + \frac{i2\pi^3}{3\sqrt{1-\mathrm{u1}}} \right. \\
& \left. -\frac{i12\pi \ln(1-u_1)}{\sqrt{1-\mathrm{u1}}} - \frac{4\pi^2 \ln(1-u_1)}{\sqrt{1-\mathrm{u1}}} - \frac{i2\pi \ln^2(1-u_1)}{\sqrt{1-\mathrm{u1}}} \right) \\
& \simeq -\frac{i\pi}{2} \ln^2(1-u_1) \cos(\phi_2 - \phi_3) \, |w| \, \ln^2|w| - \pi^2 \ln(1-u_1) \cos(\phi_2-\phi_3) \, |w| \, \ln^2|w| \\
& -i3\pi \ln(1-u_1) \cos(\phi_2 - \phi_3) \, |w| \, \ln^2|w|.
\end{aligned}$$

In (C.31) we omit terms of the order of $\ln^2|w|$ not enhanced by $\ln(1-u_1)$ because they correspond to the next-to-leading corrections in the logarithm of the energy and are irrelevant for the purpose of the present discussion. The first two terms in RHS of (C.31) reproduce the BFKL result in the Double Leading Logarithmic Approximation (DLLA) given by (26) and (27). The last term in (C.31) is currently not available in the BFKL approach and brings in some new information about the next-to-leading corrections to the BFKL eigenvalue in the adjoint representation.

D Contribution of $\gamma_1^-(p)$ up to five loops

In this section we calculate a contribution of $\gamma_1^-(p)$ to the remainder function (46) up to five loops. We show that in the Double Leading Logarithmic Approximation (DLLA) in the Mandelstam region the main contribution comes from the highest power of $\gamma_1^-(p)$ in the integral of (46), namely $(\gamma_1^-(p))^{\ell-1}$, where ℓ is a number of loops. It is useful to define

$$\gamma_1(p) = \gamma_1^+(p) + \gamma_1^-(p), \quad \gamma_1^{\pm}(p) = \psi\left(\frac{3}{2} \pm \frac{ip}{2}\right) - \psi(1) \quad \text{(D.1)}$$

and

$$h_k^-(\sigma) = \int_{-\infty}^{\infty} c^0(p) \left(\gamma_1^-(p)\right)^k e^{ip\sigma} dp. \quad \text{(D.2)}$$

We use the residue theorem noting that the function

$$f_k(p) = c^0(p) \left(\gamma_1^-(p)\right)^k e^{ip\sigma} \quad \text{(D.3)}$$

has second-order poles only at $p = \pm i$ and all other poles are simple poles. For $\sigma > 0$ we close the contour of integration in the upper semiplane, where $\gamma_1^-(p)$ has no poles. We start with $h_1^-(\sigma)$ relevant for 2 loops of the AGMSV remainder function in (46). It is easy to find the residue of $f_1(p)$ at the pole $p = i$

$$i2\pi Res(f_1(p), i) = 4e^{-\sigma}\sigma + 4e^{-\sigma} - \frac{1}{3}\pi^2 e^{-\sigma}. \tag{D.4}$$

Other poles in the upper semiplane come from $\cosh\left(\frac{\pi p}{2}\right)$ in $c^0(p)$. Using the fact that all of these poles are simple poles we can make a substitution

$$\frac{1}{\cosh\left(\frac{\pi p}{2}\right)} \to \sum_{n=1}^{\infty} \frac{i2}{\pi} \frac{(-1)^{n+1}}{p - i(2n+1)}, \tag{D.5}$$

which accounts properly for the pole and residue structure of $c^0(p)$ for $p \neq i$ in the upper semiplane. Then the calculation of the residues of $f_1(p)$ at these poles becomes straightforward and we get

$$\begin{aligned} &i2\pi \sum_{n=1}^{\infty} Res(f_1(p), i(2n+1)) = -\sum_{n=1}^{\infty} \frac{2(-1)^n e^{-\sigma-2n\sigma}(\psi(2+n) - \psi(1))}{n(1+n)} \\ &= -\sum_{n=1}^{\infty} \frac{2(-1)^n e^{-\sigma-2n\sigma} S_1(n+1)}{n(1+n)} = -4e^{-\sigma} + 4\cosh\sigma \ln\left(1 + e^{-2\sigma}\right) \\ &-2\cosh\sigma \ln^2\left(1 + e^{-2\sigma}\right) - 4\cosh\sigma \mathrm{Li}_2\left(-e^{-2\sigma}\right), \end{aligned} \tag{D.6}$$

where the we used the identity

$$S_1(n) = \psi(n+1) - \psi(1) \tag{D.7}$$

for the harmonic number $S_m(n) = \sum_{i=1}^{n} 1/i^m$ and the polygamma function $\psi(x) = d/dx(\ln\Gamma(x))$. The series summation in (D.6) as well as other sums in this section is performed using the XSummer package for FORM by Moch and Uwer [24, 27]. Finally, adding (D.6) to (D.4) we obtain

$$\begin{aligned} &h_1^-(\sigma) = \int_{-\infty}^{\infty} c^0(p)\gamma_1^-(p) e^{ip\sigma} dp = 4e^{-\sigma}\sigma - \frac{1}{3}\pi^2 e^{-\sigma} + 4\cosh\sigma \ln\left(1 + e^{-2\sigma}\right) \\ &-2\cosh\sigma \ln^2\left(1 + e^{-2\sigma}\right) - 4\cosh\sigma \mathrm{Li}_2\left(-e^{-2\sigma}\right), \end{aligned} \tag{D.8}$$

which after the analytic continuation along the path $\mathbf{B}$ results in

$$\Delta_1^-(\sigma) = -4i\pi e^{\sigma} + 8i\pi\sigma\cosh\sigma + 8i\pi\cosh\sigma\ln(2\cosh\sigma) \tag{D.9}$$

for an arbitrary positive σ, where

$$h_k^-(\sigma) \Rightarrow -h_k^-(\sigma) + \Delta_k^-(\sigma). \tag{D.10}$$

In the Regge limit ($\sigma \simeq -1/2\ln(1-u_1) \to \infty$) after the analytic continuation we obtain

$$h_1^-(\sigma) \Longrightarrow -\frac{4i\pi}{\sqrt{1-u_1}} - \frac{4i\pi\ln(1-u_1)}{\sqrt{1-u_1}} \tag{D.11}$$

and the remainder function (46) at two loops at the double logarithmic accuracy reads

$$\begin{aligned} &R_{OPE}^{(2)-} = -\cos\phi\, e^{-\tau}\, \tau h_1^-(\sigma) \Longrightarrow -i\pi\cos(\phi_2-\phi_3)|w|\ln^2(1-u_1) \\ &-i\pi\cos(\phi_2-\phi_3)|w|\ln(1-u_1) - 2i\pi\cos(\phi_2-\phi_3)|w|\ln(1-u_1)\ln|w| \\ &+2i\pi\cos(\phi_2-\phi_3)|w|\ln(2)\ln(1-u_1) - 2i\pi\cos(\phi_2-\phi_3)|w|\ln|w| + 2i\pi|w|\ln 2 \\ &\simeq -i2\pi\cos(\phi_2-\phi_3)|w|\ln|w|\ln(1-u_1) - i2\pi\cos(\phi_2-\phi_3)|w|\ln|w|. \end{aligned} \tag{D.12}$$

The expression in (D.12) reproduces the BFKL result, despite the fact that we considered only $\gamma_1^-(p)$ part (see (D.1)) of the anomalous dimension $\gamma_1(p)$ in (30).

In a similar way we calculate the contribution from $\gamma_1^-(p)$ at three loops. First we close the contour in the upper semiplane and find the residue of $f_2(p)$ in (D.3) at the second-order pole $p = i$

$$i2\pi Res(f_2(p), i) = 4e^{-\sigma}\sigma + 6e^{-\sigma} - \frac{2}{3}\pi^2 e^{-\sigma}. \tag{D.13}$$

Next we calculate the residue of $f_2(p)$ at the simple poles in the upper semiplane for $p \neq i$

$$i2\pi \sum_{n=1}^{\infty} Res(f_2(p), i(2n+1)) = -\sum_{n=1}^{\infty} \frac{2(-1)^n e^{-\sigma-2n\sigma}(\psi(2+n)-\psi(1))^2}{n(1+n)} \tag{D.14}$$
$$= -\sum_{n=1}^{\infty} \frac{2(-1)^n e^{-\sigma-2n\sigma} S_1^2(n+1)}{n(1+n)} = -2e^{-\sigma}\mathrm{Li}_2\left(-e^{-2\sigma}\right) - 4\cosh\sigma \mathrm{Li}_2\left(-e^{-2\sigma}\right)$$
$$-4\cosh\sigma \mathrm{Li}_3\left(-e^{-2\sigma}\right) + 4\cosh\sigma \ln\left(1+e^{-2\sigma}\right)\mathrm{Li}_2\left(-e^{-2\sigma}\right) - 6e^{-\sigma}$$
$$+\frac{4}{3}\cosh\sigma \ln^3\left(1+e^{-2\sigma}\right) - 4\cosh\sigma\ln^2\left(1+e^{-2\sigma}\right) + 4\cosh\sigma\ln\left(1+e^{-2\sigma}\right).$$

The expression in (D.13) together with (D.14) gives the required integral

$$h_2^-(\sigma) = \int_{-\infty}^{\infty} c^0(p)(\gamma_1^-(p))^2 e^{ip\sigma}dp = 2e^{\sigma}\mathrm{Li}_2\left(-e^{-2\sigma}\right) - 8\cosh\sigma \mathrm{Li}_2\left(-e^{-2\sigma}\right) \tag{D.15}$$
$$-4\cosh\sigma \mathrm{Li}_3\left(-e^{-2\sigma}\right) + 4\cosh\sigma\ln\left(1+e^{-2\sigma}\right)\mathrm{Li}_2\left(-e^{-2\sigma}\right) - 4e^{\sigma}\sigma + \frac{2\pi^2 e^{\sigma}}{3} + 8\sigma\cosh\sigma$$
$$-\frac{4}{3}\pi^2\cosh\sigma + \frac{4}{3}\cosh\sigma\ln^3\left(1+e^{-2\sigma}\right) - 4\cosh\sigma\ln^2\left(1+e^{-2\sigma}\right) + 4\cosh\sigma\ln\left(1+e^{-2\sigma}\right).$$

After the analytic continuation of $h_2^-(\sigma)$ along the path $\mathbf{B}$ we get

$$\Delta_2^-(\sigma) = -8i\pi\cosh\sigma \mathrm{Li}_2\left(-e^{-2\sigma}\right) - 8i\pi\sigma^2\cosh\sigma - 8i\pi e^{\sigma}\sigma - 4\pi^2 e^{\sigma} - 4i\pi e^{\sigma} \tag{D.16}$$
$$+8\pi^2\sigma\cosh\sigma + 16i\pi\sigma\cosh\sigma - \frac{4}{3}i\pi^3\cosh\sigma - 8i\pi\cosh\sigma\ln^2(2\cosh\sigma)$$
$$+8\pi^2\cosh\sigma\ln(2\cosh\sigma) + 16i\pi\cosh\sigma\ln(2\cosh\sigma).$$

In the Regge limit ($\sigma \simeq -1/2\ln(1-u_1) \to \infty$) after the analytic continuation we obtain

$$h_2^-(\sigma) \Longrightarrow -\frac{2i\pi^3}{3\sqrt{1-u_1}} - \frac{4\pi^2}{\sqrt{1-u_1}} - \frac{4i\pi}{\sqrt{1-u_1}} - \frac{2i\pi\ln^2(1-u_1)}{\sqrt{1-u_1}} \tag{D.17}$$
$$-\frac{4\pi^2\ln(1-u_1)}{\sqrt{1-u_1}} - \frac{4i\pi\ln(1-u_1)}{\sqrt{1-u_1}}$$

and the remainder function (46) at three loops in the double logarithmic approximation reads

$$R_{OPE}^{(3)-} = \cos\phi\, e^{-\tau}\frac{\tau^2}{2}h_2^-(\sigma) \Longrightarrow -\frac{i\pi}{2}\cos(\phi_2-\phi_3)|w|\ln^2|w|\ln^2(1-u_1) \tag{D.18}$$
$$-\pi^2\cos(\phi_2-\phi_3)|w|\ln^2|w|\ln(1-u_1) - i\pi\cos(\phi_2-\phi_3)|w|\ln(1-u_1).$$

For simplicity of the presentation we retain only the leading and the next-to-leading terms in $\ln|w|$ in (D.18). The first two terms in RHS of (D.18) reproduce the BFKL result in (26) and (27), while the last term is not captured by the LLA BFKL analysis.

Using the same procedure we calculate the contribution of $\gamma_1^-(p)$ at four loops. First we find the residue of $f_3(p)$ at the $p = i$

$$i2\pi Res(f_3(p), i) = e^{-\sigma}\left(8 - \pi^2 + 4\sigma\right) \tag{D.19}$$

and then at other (simple) poles in the upper semiplane

$$i2\pi \sum_{n=1}^{\infty} Res(f_3(p), i(2n+1)) = -\sum_{n=1}^{\infty} \frac{2(-1)^n e^{-\sigma-2n\sigma}(\psi(2+n)-\psi(1))^3}{n(1+n)} \tag{D.20}$$

$$= -\sum_{n=1}^{\infty} \frac{2(-1)^n e^{-\sigma-2n\sigma} S_1^3(n+1)}{n(1+n)} = -8e^{-\sigma} + \frac{4}{15}\pi^4 \cosh\sigma - e^{\sigma}\pi^2 \ln\left(1+e^{-2\sigma}\right)$$
$$+4\cosh\sigma \ln\left(1+e^{-2\sigma}\right) - 6e^{-\sigma}\sigma \ln^2\left(1+e^{-2\sigma}\right) - 6\cosh\sigma \ln^2\left(1+e^{-2\sigma}\right) + 12\sigma\cosh\sigma \ln^2\left(1+e^{-2\sigma}\right)$$
$$+2e^{-\sigma}\ln^3\left(1+e^{-2\sigma}\right) + 3e^{\sigma}\ln^3\left(1+e^{-2\sigma}\right) + 4\sigma\cosh\sigma \ln^3\left(1+e^{-2\sigma}\right) - 6e^{-\sigma}\mathrm{Li}_2\left(-e^{-2\sigma}\right)$$
$$-2e^{\sigma}\mathrm{Li}_2\left(-e^{-2\sigma}\right) + 6e^{-\sigma}\ln\left(1+e^{-2\sigma}\right)\mathrm{Li}_2\left(-e^{-2\sigma}\right) - 6\cosh\sigma \ln^2\left(1+e^{-2\sigma}\right)\mathrm{Li}_2\left(-e^{-2\sigma}\right)$$
$$-4\cosh\sigma \mathrm{Li}_2\left(-e^{-2\sigma}\right)^2 - 6e^{-\sigma}\mathrm{Li}_3\left(-e^{-2\sigma}\right) - 2e^{\sigma}\mathrm{Li}_3\left(-e^{-2\sigma}\right) + 4\cosh\sigma \ln\left(1+e^{-2\sigma}\right)\mathrm{Li}_3\left(-e^{-2\sigma}\right)$$
$$-6e^{\sigma}\mathrm{Li}_3\left(\frac{1}{1+e^{-2\sigma}}\right) - 12\cosh\sigma \ln\left(1+e^{-2\sigma}\right)\mathrm{Li}_3\left(\frac{1}{1+e^{-2\sigma}}\right) - 4\cosh\sigma \mathrm{Li}_4\left(-e^{-2\sigma}\right)$$
$$-24\cosh\sigma \mathrm{Li}_4\left(\frac{1}{1+e^{-2\sigma}}\right) + 12\cosh\sigma \mathrm{Li}_{2,2}\left(-e^{-2\sigma}\right) + 6e^{\sigma}\zeta_3 - 12\cosh\sigma \ln\left(1+e^{-2\sigma}\right)\zeta_3.$$

Adding the contributions of the second-order pole in (D.19) and simple poles in (D.20) we readily obtain

$$h_3^-(\sigma) = \int_{-\infty}^{\infty} c^0(p)\left(\gamma_1^-(p)\right)^3 e^{ip\sigma}dp = -\pi^2 e^{-\sigma} + 4\sigma e^{-\sigma} + \frac{4}{15}\pi^4\cosh\sigma - e^{\sigma}\pi^2\ln\left(1+e^{-2\sigma}\right) \tag{D.21}$$
$$+4\cosh\sigma \ln\left(1+e^{-2\sigma}\right) - 6e^{-\sigma}\sigma \ln^2\left(1+e^{-2\sigma}\right) - 6\cosh\sigma \ln^2\left(1+e^{-2\sigma}\right) + 12\sigma\cosh\sigma \ln^2\left(1+e^{-2\sigma}\right)$$
$$+2e^{-\sigma}\ln^3\left(1+e^{-2\sigma}\right) + 3e^{\sigma}\ln^3\left(1+e^{-2\sigma}\right) + 4\sigma\cosh\sigma \ln^3\left(1+e^{-2\sigma}\right) - 6e^{-\sigma}\mathrm{Li}_2\left(-e^{-2\sigma}\right)$$
$$-2e^{\sigma}\mathrm{Li}_2\left(-e^{-2\sigma}\right) + 6e^{-\sigma}\ln\left(1+e^{-2\sigma}\right)\mathrm{Li}_2\left(-e^{-2\sigma}\right) - 6\cosh\sigma \ln^2\left(1+e^{-2\sigma}\right)\mathrm{Li}_2\left(-e^{-2\sigma}\right)$$
$$-4\cosh\sigma \mathrm{Li}_2\left(-e^{-2\sigma}\right)^2 - 6e^{-\sigma}\mathrm{Li}_3\left(-e^{-2\sigma}\right) - 2e^{\sigma}\mathrm{Li}_3\left(-e^{-2\sigma}\right) + 4\cosh\sigma \ln\left(1+e^{-2\sigma}\right)\mathrm{Li}_3\left(-e^{-2\sigma}\right)$$
$$-6e^{\sigma}\mathrm{Li}_3\left(\frac{1}{1+e^{-2\sigma}}\right) - 12\cosh\sigma \ln\left(1+e^{-2\sigma}\right)\mathrm{Li}_3\left(\frac{1}{1+e^{-2\sigma}}\right) - 4\cosh\sigma \mathrm{Li}_4\left(-e^{-2\sigma}\right)$$
$$-24\cosh\sigma \mathrm{Li}_4\left(\frac{1}{1+e^{-2\sigma}}\right) + 12\cosh\sigma \mathrm{Li}_{2,2}\left(-e^{-2\sigma}\right) + 6e^{\sigma}\zeta_3 - 12\cosh\sigma \ln\left(1+e^{-2\sigma}\right)\zeta_3.$$

In a similar way we calculate also $h_4^-(\sigma)$ needed for the five-loop remainder function in (46). First we find a contribution from the second-order pole of $f_4(p)$

$$i2\pi Res(f_4(p), i) = \frac{2}{3}e^{-\sigma}\left(15 - 2\pi^2 + 6\sigma\right) \tag{D.22}$$

and then simples poles

$$i2\pi \sum_{n=1}^{\infty} Res(f_4(p), i(2n+1)) = -\sum_{n=1}^{\infty} \frac{2(-1)^n e^{-\sigma-2n\sigma}(\psi(2+n)-\psi(1))^4}{n(1+n)} \tag{D.23}$$
$$= -\sum_{n=1}^{\infty} \frac{2(-1)^n e^{-\sigma-2n\sigma} S_1^4(n+1)}{n(1+n)}.$$

The series in (D.23) is also summed using the XSummer package for FORM and together with the double-pole contribution in (D.22) it results in

$$
\begin{aligned}
h_4^-(\sigma) &= \int_{-\infty}^{\infty} c^0(p)\left(\gamma_1^-(p)\right)^4 e^{ip\sigma} dp = \frac{4}{15}e^{-\sigma}\pi^4 - \frac{4}{3}e^{-\sigma}\pi^2 + 16e^{\sigma}\zeta_3 \qquad \text{(D.24)}\\
&-\frac{6}{5}\cosh\sigma \ln^5\left(1+e^{-2\sigma}\right) - 3e^{-\sigma}\ln^4\left(1+e^{-2\sigma}\right) - 8\sigma\cosh\sigma\ln^4\left(1+e^{-2\sigma}\right)\\
&+6\cosh\sigma\ln^4\left(1+e^{-2\sigma}\right) - \frac{8}{3}e^{-\sigma}\ln^3\left(1+e^{-2\sigma}\right) - 8e^{-\sigma}\sigma\ln^3\left(1+e^{-2\sigma}\right)\\
&+32\sigma\cosh\sigma\ln^3\left(1+e^{-2\sigma}\right) + \frac{40}{3}\cosh\sigma\ln^3\left(1+e^{-2\sigma}\right) + 8\cosh\sigma\mathrm{Li}_2\left(-e^{-2\sigma}\right)\ln^3\left(1+e^{-2\sigma}\right)\\
&+16e^{\sigma}\sigma\ln^2\left(1+e^{-2\sigma}\right) - 8\cosh\sigma\ln^2\left(1+e^{-2\sigma}\right) - 24\cosh\sigma\mathrm{Li}_2\left(-e^{-2\sigma}\right)\ln^2\left(1+e^{-2\sigma}\right)\\
&+12e^{\sigma}\mathrm{Li}_2\left(\frac{1}{1+e^{-2\sigma}}\right)\ln^2\left(1+e^{-2\sigma}\right) - 8\cosh\sigma\mathrm{Li}_3\left(-e^{-2\sigma}\right)\ln^2\left(1+e^{-2\sigma}\right)\\
&+24\cosh\sigma\mathrm{Li}_3\left(\frac{1}{1+e^{-2\sigma}}\right)\ln^2\left(1+e^{-2\sigma}\right) + 24\cosh\sigma\zeta_3\ln^2\left(1+e^{-2\sigma}\right)\\
&+4\cosh\sigma\mathrm{Li}_2\left(-e^{-2\sigma}\right)^2\ln\left(1+e^{-2\sigma}\right) - \frac{8}{15}\pi^4\cosh\sigma\ln\left(1+e^{-2\sigma}\right) - \frac{16}{3}\pi^2\cosh\sigma\ln\left(1+e^{-2\sigma}\right)\\
&+4\cosh\sigma\ln\left(1+e^{-2\sigma}\right) + 16e^{-\sigma}\mathrm{Li}_2\left(-e^{-2\sigma}\right)\ln\left(1+e^{-2\sigma}\right) - 8\cosh\sigma\mathrm{Li}_2\left(-e^{-2\sigma}\right)\ln\left(1+e^{-2\sigma}\right)\\
&+16\cosh\sigma\mathrm{Li}_3\left(-e^{-2\sigma}\right)\ln\left(1+e^{-2\sigma}\right) - 24e^{-\sigma}\mathrm{Li}_3\left(\frac{1}{1+e^{-2\sigma}}\right)\ln\left(1+e^{-2\sigma}\right)\\
&+4\cosh\sigma\mathrm{Li}_4\left(-e^{-2\sigma}\right)\ln\left(1+e^{-2\sigma}\right) + 48\cosh\sigma\mathrm{Li}_4\left(\frac{1}{1+e^{-2\sigma}}\right)\ln\left(1+e^{-2\sigma}\right)\\
&-32\cosh\sigma\mathrm{Li}_{2,2}\left(-e^{-2\sigma}\right)\ln\left(1+e^{-2\sigma}\right) - 48\cosh\sigma\zeta_3\ln\left(1+e^{-2\sigma}\right) + \frac{8}{3}e^{-\sigma}\pi^2\ln\left(1+e^{-2\sigma}\right)\\
&-6e^{-\sigma}\mathrm{Li}_2\left(-e^{-2\sigma}\right)^2 - 4\cosh\sigma\mathrm{Li}_2\left(-e^{-2\sigma}\right)^2 + 4e^{-\sigma}\sigma + \frac{8}{15}\pi^4\cosh(\sigma)\\
&-8\cosh\sigma H_{0,0,0,1,1}\left(-e^{-2\sigma}\right) - 20\cosh\sigma H_{0,0,1,0,1}\left(-e^{-2\sigma}\right) - 48\cosh\sigma H_{0,0,0,1,1}\left(-e^{-2\sigma}\right)\\
&-12\cosh\sigma H_{0,1,0,0,1}\left(-e^{-2\sigma}\right) - 16\cosh\sigma H_{0,1,0,1,1}\left(-e^{-2\sigma}\right) - 16\cosh\sigma H_{0,1,1,0,1}\left(-e^{-2\sigma}\right)\\
&-6e^{-\sigma}\mathrm{Li}_2\left(-e^{-2\sigma}\right) - 4\cosh(\sigma)\mathrm{Li}_2\left(-e^{-2\sigma}\right) - 10e^{-\sigma}\mathrm{Li}_3\left(-e^{-2\sigma}\right)\\
&-4\cosh\sigma\mathrm{Li}_3\left(-e^{-2\sigma}\right) - 16e^{\sigma}\mathrm{Li}_3\left(\frac{1}{1+e^{-2\sigma}}\right) - 6e^{-\sigma}\mathrm{Li}_4\left(-e^{-2\sigma}\right)\\
&-4\cosh\sigma\mathrm{Li}_4\left(-e^{-2\sigma}\right) + 24e^{\sigma}\mathrm{Li}_4\left(\frac{1}{1+e^{-2\sigma}}\right) - 96\cosh\sigma\mathrm{Li}_4\left(\frac{1}{1+e^{-2\sigma}}\right)\\
&-4\cosh\sigma\mathrm{Li}_5\left(-e^{-2\sigma}\right) + 8e^{-\sigma}\mathrm{Li}_{2,2}\left(-e^{-2\sigma}\right) + 32\cosh\sigma\mathrm{Li}_{2,2}\left(-e^{-2\sigma}\right).
\end{aligned}
$$

The function $h_4^-(\sigma)$ in (D.24) is expressed in terms of the classical polylogarithms, and the harmonic polylogarithms (HPL) [24] $H_{a_1,a_2,\ldots,a_n}(x)$ are recursively defined by

$$H_{a_1,a_2,\ldots,a_n}(x) = \int_0^x dt g_{a_1}(t) H_{a_2,\ldots,a_n}(t), \quad a_i = 0, \pm 1, \qquad \text{(D.25)}$$

where

$$g_{\pm}(x) = \frac{1}{1 \mp x}, \quad g_0(x) = \frac{1}{x}, \quad H_{\pm}(x) = \mp \ln(1 \mp x), \quad H_0(x) = \ln x \qquad \text{(D.26)}$$

and at least one of the indices a_i is not zero. For all $a_i = 0$, one has

$$H_{0,0,\ldots,0}(x) = \frac{1}{n!}\ln^n x. \qquad \text{(D.27)}$$

For a nice introduction and the list of HPL with the transcendentality (number of indices a_i) up to four the reader is referred to the appendix of [28]. In particular, we used here

$$H_{0,0,1,1}(x) = \mathrm{Li}_{2,2}(x). \qquad \text{(D.28)}$$

The analytic continuation is similar to that done for two and three loops though due to a complexity of the calculations it is much easier to perform the continuation together with subsequent Regge limit $\sigma \to +\infty$. In this case only those HPL, which have all rightmost indices 1 contribute in (D.24). Other HPL, where the index 0 stand to right of the index 1 all vanish for $\sigma \to +\infty$ after the analytic continuation. As simple example, we analytically continue $H_{1,0,1}\left(-e^{-2\sigma}\right)$, noting that

$$-e^{-2\sigma} \simeq -\frac{1-u_1}{u_1} \tag{D.29}$$

for the analytic continuation along the path $\mathbf{B}$, where $\sigma \simeq 1/2 \ln(u_1/(1-u_1))$. In the multi-Regge kinematics $|u_1| \to 1^-$ and the expression in (D.29) rotates in the anti-clockwise direction around the origin as explained in the previous section. Using the integral representation of HPL (D.25) we perform the analytic continuation

$$H_{1,0,1}\left(-e^{-2\sigma}\right) = \int_0^{-\frac{1-u_1}{u_1}} \frac{dt}{1-t} \int_0^t \frac{dt'}{t'} \int_0^{t'} \frac{dt''}{1-t''} \Rightarrow \tag{D.30}$$

$$\int_0^{-\frac{1-|u_1|}{|u_1|}} \frac{dt}{1-t} \int_1^t \frac{dt'}{t'} - i2\pi \int_0^{-\frac{1-|u_1|}{|u_1|}} \frac{dt'}{t'} \int_0^{t'} \frac{dt''}{1-t''} - i2\pi \int_0^{-\frac{1-|u_1|}{|u_1|}} \frac{dt}{1-t} \int_1^t \frac{dt'}{t'}$$

$$= H_{1,0,1}\left(-e^{-2\sigma}\right) - i2\pi H_{0,1}\left(-e^{-2\sigma}\right) - i2\pi H_{1,0}\left(-e^{-2\sigma}\right) \overset{\text{Regge limit}}{\longrightarrow} 0. \tag{D.31}$$

Note that the singularities of HPL functions are determined by the rightmost index. All HPL appearing in (D.24) have the same rightmost index 1. This means that the branch cut of all HPL appearing in (D.24) is the same as the branch cut of $H_1(-e^{-2\sigma}) = -\ln\left(1+e^{-2\sigma}\right)$. The classical polylogarithms $\mathrm{Li}_n\left(-e^{-2\sigma}\right)$ have the same cut structure as well, because they are a special case of HPL with the rightmost index 1, namely

$$\mathrm{Li}_n\left(-e^{-2\sigma}\right) = H_{0,0,\ldots,0,1}\left(-e^{-2\sigma}\right), \tag{D.32}$$

where n is the number of indices. For $\mathrm{Li}_n\left(-e^{-2\sigma}\right)$ all but the rightmost indices are 0, so that they do make a contribution after the analytic continuation in the Regge limit and can be compactly written as

$$\mathrm{Li}_n\left(-e^{-2\sigma}\right) \Rightarrow \mathrm{Li}_n\left(-e^{-2\sigma}\right) - i2\pi \frac{(-2\sigma+i\pi)^{n-1}}{(n-1)!} \tag{D.33}$$

for arbitrary value of $\sigma > 0$. In the expressions for $h_3(\sigma)$ and $h_4(\sigma)$ for the first time we have the generalized Nielsen polylogarithm $\mathrm{Li}_{2,2}\left(-e^{-2\sigma}\right)$. We write the integral representation of the corresponding HPL and analytically continue it (along the path $\mathbf{B}$) to our region

$$\mathrm{Li}_{2,2}\left(-e^{-2\sigma}\right) = H_{0,0,1,1}\left(-e^{-2\sigma}\right) = \int_0^{-\frac{1-u_1}{u_1}} \frac{dt}{t} \int_0^t \frac{dt'}{t'} \int_0^{t'} \frac{dt''}{1-t''} \int_0^{t''} \frac{dt'''}{1-t'''} \tag{D.34}$$

$$\int_0^{-\frac{1-u_1}{u_1}} \frac{dt}{t} \int_0^t \frac{dt'}{t'} \frac{1}{2} \ln^2(1-t') \Rightarrow \int_0^{-\frac{1-|u_1|}{|u_1|}} \frac{dt}{t} \int_0^t \frac{dt'}{t'} \frac{1}{2} \left(\ln(1-t') + i2\pi\right)^2$$

$$= H_{0,0,1,1}\left(-e^{-2\sigma}\right) - i2\pi H_{0,0,1}\left(-e^{-2\sigma}\right) + \frac{(i2\pi)^2}{2} H_{0,0}\left(-e^{-2\sigma}\right) \overset{\text{Regge limit}}{\longrightarrow} \frac{(i2\pi)^2}{2} \frac{1}{2} (-2\sigma+i\pi)^2 .$$

There are also classical polylogarithms of the argument

$$\frac{1}{1+e^{-2\sigma}} \simeq u_1 \tag{D.35}$$

in $h_3(\sigma)$ and $h_4(\sigma)$. However, they remain the same after the analytic continuation because the circular path of the continuation $u_1 = |u_1|e^{i\psi}$ with $0 \le \psi \le -i2\pi$ never crosses the singularities of these functions. Using these results we readily find the contribution of the maximal powers of $\gamma_1^-(p)$ in the integrand of the remainder function (46). In the Mandestam region in the multi-Regge kinematics we get

$$h_3^-(\sigma) \Longrightarrow \frac{24i\pi\zeta_3}{\sqrt{1-u_1}} - \frac{8\pi^4}{3\sqrt{1-u_1}} + \frac{4i\pi^3}{\sqrt{1-u_1}} - \frac{8\pi^2}{\sqrt{1-u_1}} - \frac{4i\pi}{\sqrt{1-u_1}} - \frac{2i\pi\ln^3(1-u_1)}{3\sqrt{1-u_1}} \tag{D.36}$$
$$-\frac{4\pi^2\ln^2(1-u_1)}{\sqrt{1-u_1}} - \frac{2i\pi\ln^2(1-u_1)}{\sqrt{1-u_1}} + \frac{6i\pi^3\ln(1-u_1)}{\sqrt{1-u_1}} - \frac{8\pi^2\ln(1-u_1)}{\sqrt{1-u_1}} - \frac{4i\pi\ln(1-u_1)}{\sqrt{1-u_1}},$$

which gives the four loop expression

$$R_{OPE}^{(4)-} = -\cos\phi\, e^{-\tau}\,\frac{\tau^3}{3!}h_3^-(\sigma) \Longrightarrow -\frac{i\pi}{18}\cos(\phi_2-\phi_3)|w|\ln^3|w|\ln^3(1-u_1) \tag{D.37}$$
$$-\frac{\pi^2}{3}\cos(\phi_2-\phi_3)|w|\ln^3|w|\ln^2(1-u_1) - \frac{i\pi}{6}\cos(\phi_2-\phi_3)|w|\ln^3|w|\ln^2(1-u_1).$$

In (D.37) we retain only the leading and next-to-leading powers of $\ln(1-u_1)$ in terms leading in $\ln|w|$. At five loops for $\sigma\to\infty$ we obtain

$$h_4^-(\sigma) \Longrightarrow \frac{48\pi^2\zeta_3}{\sqrt{1-u_1}} + \frac{48i\pi\zeta_3}{\sqrt{1-u_1}} - \frac{16\zeta_3}{\sqrt{1-u_1}} - \frac{353i\pi^5}{6\sqrt{1-u_1}} + \frac{678\pi^4}{5\sqrt{1-u_1}} - \frac{154i\pi^3}{3\sqrt{1-u_1}} \tag{D.38}$$
$$-\frac{12\pi^2}{\sqrt{1-u_1}} - \frac{4i\pi}{\sqrt{1-u_1}} - \frac{i\pi\ln^4(1-u_1)}{6\sqrt{1-u_1}} - \frac{2\pi^2\ln^3(1-u_1)}{\sqrt{1-u_1}} - \frac{2i\pi\ln^3(1-u_1)}{3\sqrt{1-u_1}} + \frac{9i\pi^3\ln^2(1-u_1)}{\sqrt{1-u_1}}$$
$$-\frac{6\pi^2\ln^2(1-u_1)}{\sqrt{1-u_1}} - \frac{2i\pi\ln^2(1-u_1)}{\sqrt{1-u_1}} + \frac{22\pi^4\ln(1-u_1)}{3\sqrt{1-u_1}} + \frac{18i\pi^3\ln(1-u_1)}{\sqrt{1-u_1}} - \frac{12\pi^2\ln(1-u_1)}{\sqrt{1-u_1}}$$
$$-\frac{4i\pi\ln(1-u_1)}{\sqrt{1-u_1}} \simeq -\frac{i\pi\ln^4(1-u_1)}{6\sqrt{1-u_1}} - \frac{2\pi^2\ln^3(1-u_1)}{\sqrt{1-u_1}} - \frac{2i\pi\ln^3(1-u_1)}{3\sqrt{1-u_1}},$$

which gives

$$R_{OPE}^{(5)-} = \cos\phi\, e^{-\tau}\,\frac{\tau^4}{4!}h_4^-(\sigma) \Longrightarrow -\frac{i\pi}{288}\cos(\phi_2-\phi_3)|w|\ln^4|w|\ln^4(1-u_1) \tag{D.39}$$
$$-\frac{\pi^2}{24}\cos(\phi_2-\phi_3)|w|\ln^4|w|\ln^3(1-u_1) - \frac{i\pi}{72}\cos(\phi_2-\phi_3)|w|\ln^4|w|\ln^3(1-u_1).$$

As in the previous case we leave only the leading and the next-to-leading powers of $\ln(1-u_1)$ in terms leading in $\ln|w|$. The first terms in RHS of both (D.37) and (D.39) reproduce the BFKL result in the Double Leading Logarithmic Approximation given by (26) and (27). The last term in RHS of both (D.37) and (D.39) is currently not available in the LLA BFKL approach and requires a knowledge of the next-to-leading BFKL intercept in the adjoint representation. We showed that the BFKL prediction in the Double Leading Logarithmic Approximation can be reproduced up to five loops by analytically continuing the OPE expression for the remainder function (46) to the Mandelstam region, taking into account only the maximal powers of $\gamma_1^-(p)$.

On the other hand any power of $\gamma_1^+(p)$ in the integrand of (46) makes the corresponding part of the remainder function to be suppressed by one power of $\ln(1-u_1)$ in the Mandelstam region. These subleading contribution are not captured by the BFKL analysis presented here and require a knowledge of the next-to-leading BFKL intercept in the adjoint representation as it has been mentioned before. As a simple example of the above statement, we calculate the contribution of $\gamma_1^+(p)$ at two and three loops. For this purpose we generalize the definition of $h_k(\sigma)$ in (D.2) to include also powers of $\gamma_1^+(p)$ in the integrand

$$h_k^{-,\ldots,+}(\sigma) = \int_{-\infty}^{\infty} c^0(p)\left(\gamma_1^-(p)\right)^m \left(\gamma_1^+(p)\right)^{k-m} e^{ip\sigma}dp. \tag{D.40}$$

From the corresponding expression for $\gamma_1(p) = \gamma_1^+(p) + \gamma_1^-(p)$ given by $h_1(\sigma)$ in (44) and $h_1^-(\sigma)$ in (D.8) we readily obtain

$$h_1^+(\sigma) = h_1(\sigma) - h_1^-(\sigma) = 4\cosh\sigma \mathrm{Li}_2\left(-e^{-2\sigma}\right) + 6\sigma^2\cosh\sigma + 4e^{-\sigma}\sigma + \frac{1}{3}\pi^2 e^{-\sigma} \tag{D.41}$$
$$-4\sigma\cosh\sigma - 2\cosh\sigma\ln^2(2\cosh\sigma) - 4\sigma\cosh\sigma\ln(2\cosh\sigma) + 4\cosh\sigma\ln(2\cosh\sigma),$$

which after the analytic continuation along the path **B** gives

$$h_1^+(\sigma) \Rightarrow -h_1^+(\sigma) + \Delta_1^+(\sigma), \ \ \Delta_1^+(\sigma) = -4i\pi e^{\sigma}. \tag{D.42}$$

The leading contribution of the remainder function (46) in the Mandelstam channel then is given by

$$R_{OPE}^{(2)+} = -\cos\phi e^{-\tau}\tau h_1^+(\sigma) \Longrightarrow -i\pi\cos(\phi_2-\phi_3)|w|\ln(1-u_1) \tag{D.43}$$
$$-i2\pi\cos(\phi_2-\phi_3)|w|\ln|w| + i2\pi\cos(\phi_2-\phi_3)|w|\ln 2,$$

which is suppressed by at least one power of $\ln(1-u_1)$ with respect to the leading term in $R_{OPE}^{(2)-}$ of (D.12).

This suppression holds also at three loops as described below. Using $h_2^-(\sigma) = h_2^{--}(\sigma)$ in (D.15) we can easily obtain

$$h_2^{++}(\sigma) = \int_{-\infty}^{\infty} c^0(p)(\gamma_1^+(p))^2 e^{ip\sigma}dp \tag{D.44}$$

by complex conjugation of $h_2^-(\sigma)$ with subsequent substitution $\sigma \to -\sigma$

$$h_2^{++}(\sigma) = h_2^{--}(-\sigma) = -2e^{-\sigma}\mathrm{Li}_2\left(-e^{-2\sigma}\right) + 8\mathrm{Li}_2\left(-e^{-2\sigma}\right)\cosh\sigma - 4\mathrm{Li}_3\left(-e^{-2\sigma}\right)\cosh\sigma \tag{D.45}$$
$$-4\mathrm{Li}_2\left(-e^{-2\sigma}\right)\ln\left(1+e^{2\sigma}\right)\cosh\sigma + \frac{16}{3}\sigma^3\cosh\sigma - 4e^{-\sigma}\sigma^2 + 16\sigma^2\cosh\sigma$$
$$-8\sigma^2\ln\left(1+e^{2\sigma}\right)\cosh(\sigma) + 4e^{-\sigma}\sigma + \frac{1}{3}\pi^2 e^{-\sigma} + \frac{4}{3}\pi^2\sigma\cosh\sigma - 8\sigma\cosh\sigma$$
$$+\frac{4}{3}\ln^3\left(1+e^{2\sigma}\right)\cosh\sigma - 4\ln^2\left(1+e^{2\sigma}\right)\cosh\sigma - \frac{2}{3}\pi^2\ln\left(1+e^{2\sigma}\right)\cosh\sigma + 4\ln\left(1+e^{2\sigma}\right)\cosh\sigma.$$

The mixed term $h_2^{+-}(\sigma)$ defined by

$$h_2^{+-}(\sigma) = \int_{-\infty}^{\infty} c^0(p)\gamma_1^+(p)\gamma_1^-(p)e^{ip\sigma}dp \tag{D.46}$$

can be readily obtained from $h_2(\sigma)$ in (52), $h_2^-(\sigma) = h_2^{--}(\sigma)$ in (D.15) and $h_2^{++}(\sigma)$ in (D.45)

$$h_2^{+-}(\sigma) = \frac{1}{2}\left(h_2(\sigma) - h_2^{++}(\sigma) - h_2^{--}(\sigma)\right) = 8\sigma\mathrm{Li}_2\left(-e^{-2\sigma}\right)\cosh\sigma \tag{D.47}$$
$$+\frac{16}{3}\sigma^3\cosh\sigma + 8e^{-\sigma}\sigma - \frac{4}{3}\pi^2\sigma\cosh\sigma - 16\sigma\cosh\sigma + \frac{4}{3}\ln^3\left(1+e^{2\sigma}\right)\cosh\sigma$$
$$-4\sigma\ln^2\left(1+e^{2\sigma}\right)\cosh\sigma - 4\ln^2\left(1+e^{2\sigma}\right)\cosh\sigma + 8\sigma\ln\left(1+e^{2\sigma}\right)\cosh\sigma$$
$$+\frac{2}{3}\pi^2\ln\left(1+e^{2\sigma}\right)\cosh\sigma + 8\ln\left(1+e^{2\sigma}\right)\cosh\sigma + 8\mathrm{Li}_3\left(-e^{-2\sigma}\right)\cosh\sigma.$$

Next we perform the analytic continuation of $h_2^{+-}(\sigma)$ and $h_2^{++}(\sigma)$ along path **B** obtaining

$$h_2^{++}(\sigma) \Rightarrow -h_2^{++}(\sigma) + \Delta_2^{++}(\sigma), \ \ \Delta_2^{++}(\sigma) = -4i\pi e^{\sigma} \tag{D.48}$$

and

$$h_2^{+-}(\sigma) \Rightarrow -h_2^{+-}(\sigma) + \Delta_2^{+-}(\sigma), \tag{D.49}$$

where

$$\Delta_2^{+-}(\sigma) = 8i\pi\mathrm{Li}_2\left(-e^{-2\sigma}\right)\cosh\sigma + 16i\pi\sigma^2\cosh\sigma + 8i\pi e^{-\sigma} + \frac{4}{3}i\pi^3\cosh\sigma \quad \text{(D.50)}$$
$$-16i\pi\cosh\sigma - 4i\pi\ln^2\left(1+e^{2\sigma}\right)\cosh\sigma + 8i\pi\ln\left(1+e^{2\sigma}\right)\cosh\sigma.$$

In the Regge limit $\sigma \to \infty$ this reads

$$h_2^{++}(\sigma) \Longrightarrow -\frac{4i\pi}{\sqrt{1-u_1}},\;\; h_2^{+-}(\sigma) \Longrightarrow \frac{2i\pi^3}{3\sqrt{1-u_1}} - \frac{8i\pi}{\sqrt{1-u_1}} - \frac{4i\pi\ln(1-u_1)}{\sqrt{1-u_1}}. \quad \text{(D.51)}$$

Finally plugging (D.51) in $R_{OPE}^{(3)++}$ and $R_{OPE}^{(3)+-}$ defined by

$$R_{OPE}^{(3)++} = \cos\phi\, e^{-\tau}\frac{\tau^2}{2}h_2^{++}(\sigma),\;\; R_{OPE}^{(3)+-} = \cos\phi\, e^{-\tau}\tau^2 h_2^{+-}(\sigma), \quad \text{(D.52)}$$

we obtain

$$R_{OPE}^{(3)+-} \Longrightarrow \frac{i\pi}{12}\cos(\phi_2-\phi_3)|w|\left(-6\ln(1-u_1)+\pi^2-12\right)\left(\ln(1-u_1)+2\ln|w|-2\ln 2\right)^2$$
$$\simeq -i2\pi\cos(\phi_2-\phi_3)|w|\ln(1-u_1)\ln^2|w| + \frac{i\pi^3}{3}\cos(\phi_2-\phi_3)|w|\ln^2|w| \quad \text{(D.53)}$$
$$-i4\pi\cos(\phi_2-\phi_3)|w|\ln^2|w|$$

and

$$R_{OPE}^{(3)++} \Longrightarrow -i\frac{\pi}{4}\cos(\phi_2-\phi_3)|w|\ln^2(1-u_1) \quad \text{(D.54)}$$
$$+i\pi\cos(\phi_2-\phi_3)|w|\ln 2\ln(1-u_1) - i\pi|w|\ln^2|w| - i\pi\cos(\phi_2-\phi_3)|w|\ln^2 2$$
$$+i2\pi\cos(\phi_2-\phi_3)|w|\ln 2\ln|w| - i\pi\cos(\phi_2-\phi_3)|w|\ln(1-u_1)\ln|w| \simeq -i\pi|w|\ln^2|w|.$$

Comparing (D.18), (D.54) and (D.53) one can see that each power of $\gamma_1^+(p)$ in the integrand of the remainder function R_{OPE} in (46) introduces an additional suppression by one power of $\ln(1-u_1)$ in the terms leading in $\ln|w|$.

In this section we found that the BFKL result in the Double Leading Logarithmic Approximation (DLLA) can be reproduced taking into account only $\gamma_1^-(p)$ in the OPE remainder function (46). Each power of $\gamma_1^+(p)$ introduce an additional suppression in $\ln(1-u_1)$ and the corresponding contributions are not captured by the LLA BFKL analysis.

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